THE

CAMBRIDGE

MATHEMATICAL JOURNAL.

Sent P. Kelland Sep
1845

VOL. IV.

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Οὐδὲ μὲν οὐδ' οἱ ἄναρχοι ἔσαν, πόθεόν γε μὲν ἀρχόν.

CAMBRIDGE:
PUBLISHED BY E. JOHNSON, TRINITY STREET;
AND WHITTAKER & CO., LONDON.

1845

CAMBRIDGE.
PRINTED BY METCALFE AND PALMER, TRINITY-STREET.

THE CAMBRIDGE

MATHEMATICAL JOURNAL.

VOL. IV.] NOVEMBER, 1843. [No. XIX.

I.—ON THE EVALUATION OF DEFINITE MULTIPLE INTEGRALS.

By R. L. ELLIS, M.A., Fellow of Trinity College.

THE following pages contain some general results obtained by means of Fourier's theorem. A few words will be sufficient to explain the manner in which it has been applied.

A definite multiple integral, where the limits are given by the inequality

$$f(xy\ldots.) \overset{=}{>} h_1 \overset{=}{<} h,$$

may be treated as if the limits of the different variables were independent of one another, provided the function under the signs of integration be considered discontinuous, and equal to zero whenever $f(xy\ldots.)$ transgresses the assigned limits h_1 and h. This idea has been made use of by M. Lejeune Dirichlet. It had, however, occurred to me before I was acquainted with his paper on multiple integrals, and the way in which I have applied it is, I believe, new.

Suppose the function to be integrated were of the form $\phi\,(xy\ldots.)\,\psi\,\{f(xy\ldots.)\}$: the limits being given by the inequality already mentioned. Then, by Fourier's theorem, writing $f.$ for $f(xy\ldots.)$,

$$\psi f. = \frac{1}{\pi}\int_0^\infty da \int_{h_1}^{h} \psi u.\ \cos a\,(f. - u)\,du,$$

for all values of $f.$ which lie between h_1 and h; moreover for the purposes of integration the formula may be applied, except in particular cases, so as to include these limiting values. (See Poisson, *Théorie de la Chaleur*); while for all

values of $f.$ which lie without these limits, the second side of the equation is equal to zero. Consequently the integral

$$\int dx \int dy \ldots \phi(xy\ldots)\,\psi\{f(xy\ldots)\}$$

$$= \frac{1}{\pi}\int_{h1}^{h} \psi u du \int_0^\infty da \int dx \int dy \ldots \phi(xy\ldots) \cos a(f. - u) \ldots\ldots (1);$$

the limits on the first side being given in the manner already mentioned. Those of the integrations with respect to $x, y,$ &c. are arbitrary, provided they include all values of the variables which satisfy the given inequality.

Again, if in the given multiple integral the limits were determined by the single relation $f. \leqq h$, joined to the conditions that $x, y,$ &c. were to have no values less than certain assigned limits; *e.g.* if we were to consider only positive values of the variables, the formula (1) would still apply with a slight modification. The inferior limit of integration with respect to u would be arbitrary, provided it included all the values which could be given to $f.$ by admissible values of the variables, while the inferior limits of integration for $x, y,$ &c. would be determined by the particular conditions of the case.

Let us take, as an example of the method, the integral

$$\int dx \int dy \ldots x^{a-1} y^{b-1} z^{c-1} \ldots f(mx + ny + \ldots) \ldots\ldots (A),$$

$m, n,$ &c. being all positive, and the limits being given by

$$mx + ny + \ldots \leqq h,$$

no negative values of the variables being admitted. In this case (1) becomes

$$\int dx \int dy \ldots x^{a-1} y^{b-1} \ldots f(mx + ny + \ldots)$$

$$= \frac{1}{\pi}\int_{h1}^{h} fudu \int_0^\infty da \int_0 dx \int_0 dy \ldots x^{a-1} y^{b-1} \ldots \cos a(mx+ny+\ldots-u) \ldots (2),$$

and the integrations with respect to $x, y,$ &c. may be conveniently extended to infinity; (h_1 it should be observed is an arbitrary quantity < 0).

I first seek the value of

$$\int_0^\infty dx \int_0^\infty dy \ldots x^{a-1} y^{b-1} \ldots \cos a(mx + ny + \ldots - u) = (B).$$

Integrating first for x, we get

$$(B) = \frac{\Gamma(a)}{m^a a^a}\int_0^\infty dy \ldots y^{b-1} \ldots \cos\left\{a\frac{\pi}{2} + a(ny + \ldots - u)\right\}.$$

This follows from the formulæ

$$\left.\begin{aligned}\int_0^\infty x^{a-1}\cos \alpha x dx &= \frac{\Gamma(a)}{\alpha^a}\cos a\frac{\pi}{2}\\ \int_0^\infty x^{a-1}\sin \alpha x dx &= \frac{\Gamma(a)}{\alpha^a}\sin a\frac{\pi}{2}\end{aligned}\right\}\ldots\ldots(\gamma).$$

Integrating in a similar manner for y, &c. successively, we get ultimately

$$(\mathrm{B}) = \frac{\Gamma(a)\,\Gamma(b)\ldots}{m^a n^b\ldots\alpha^{a+b+\ldots}}\cos\left\{(a+b+\ldots)\frac{\pi}{2} - \alpha u\right\}.$$

Now by (γ), we easily see that

$$\int_0^\infty v^{a+b+\ldots-1}\cos\alpha(v-u)\,dv = \frac{\Gamma(a+b+\ldots)}{\alpha^{a+b+\ldots}}\cos\left\{(a+b+\ldots)\frac{\pi}{2} - \alpha u\right\},$$

and therefore

$$(\mathrm{B}) = \frac{1}{m^a n^b\ldots\ldots}\cdot\frac{\Gamma(a)\,\Gamma(b)\ldots\ldots}{\Gamma(a+b+\ldots\ldots)}\int_0^\infty v^{a+b+\ldots\ldots-1}\cos\alpha(v-u)\,dv.$$

Hence $\displaystyle\int_0^\infty d\alpha\,(\mathrm{B}) = \frac{\pi}{m^a n^b\ldots\ldots}\cdot\frac{\Gamma(a)\,\Gamma(b)\ldots\ldots}{\Gamma(a+b+\ldots\ldots)}\,u^{a+b+\ldots\ldots-1}$,

for all positive values of u, and $= 0$ for all negative values, by Fourier's theorem. Consequently the second side of (2) becomes

$$\frac{1}{m^a n^b\ldots\ldots}\;\frac{\Gamma(a)\,\Gamma(b)\ldots\ldots}{\Gamma(a+b+\ldots\ldots)}\int_0^h fu u^{a+b+\ldots\ldots-1}du + \frac{1}{\pi}\int_{h_1}^0 fu.0.du.$$

Thus finally

$$\int dx\int dy\ldots\ldots x^{a-1}\,y^{b-1}\ldots\ldots f(mx+ny+\ldots\ldots)$$
$$= \frac{1}{m^a n^b\ldots\ldots}\;\frac{\Gamma(a)\,\Gamma(b)\ldots\ldots}{\Gamma(a+b+\ldots\ldots)}\int_0^h fu\,u^{a+b+\ldots\ldots-1}\,du\ \ldots\ldots(3);$$

which is equivalent to Liouville's extension of Dirichlet's theorem.

I proceed to evaluate the definite integral

$$\int_0 dx\int_0 dy\ldots\ldots e^{-(max+nby+\ldots)}f(mx+ny+\ldots\ldots)\ldots\ldots(\mathrm{E}),$$

$ab\ldots\ldots$ and $mn\ldots\ldots$ being all positive, and the limits being given by

$$mx+ny+\ldots\ldots \leqq h.$$

By the general formula

$$E = \frac{1}{\pi}\int_{h_1}^h fu\,du\int_0^\infty d\alpha\int_0^\infty dx\int_0^\infty dy\ldots e^{-(max+nby+\ldots)}\cos\alpha(mx+ny+\ldots-u)$$
$$(h_1 < 0).$$

Let $F = \int_0^\infty dx \int_0^\infty dy \ldots e^{-(max+nby+\ldots)} \cos \alpha\,(mx + ny + \ldots - u)$;

then $F = \cos \alpha u \int_0^\infty dx \int_0^\infty dy \ldots e^{-(max+nby+\ldots)} \cos \alpha\,(mx + ny + \ldots)$

$+ \sin \alpha u \int_0^\infty dx \int_0^\infty dy \ldots e^{-(max+nby+\ldots)} \sin \alpha\,(mx + ny + \ldots)$;

which we may put equal to $G \cos \alpha u + H \sin \alpha u$.

First to find the value of G

$$= \int_0^\infty dx \int_0^\infty dy \ldots e^{-(max+nby+\ldots)} \cos \alpha\,(mx + ny + \ldots).$$

Develope the cosine; the result is composed of terms, each containing sines or cosines of all the variables.

Also by the formulæ

$$\left.\begin{aligned} \int_0^\infty e^{-max} \cos \alpha\, mx\, dx &= \frac{a}{m(a^2 + \alpha^2)} \\ \int_0^\infty e^{-max} \sin \alpha\, mx\, dx &= \frac{\alpha}{m(a^2 + \alpha^2)} \end{aligned}\right\} \ldots\ldots\ldots\ldots (\,);$$

we see that every factor whether sine or cosine introduces on integration a factor of the form $\frac{1}{m(a^2 + \alpha^2)}$. Moreover a sine factor introduces α in the numerator, a cosine factor a or b or &c.

Let P represent the continued product of a, b, c &c. and D that of $m(a^2 + \alpha^2)\ \ n(b^2 + \alpha^2)$ &c. Then $G = \frac{P}{D} \Sigma C \frac{\alpha^\mu}{ab\ldots}$, where μ is a positive integer less than the whole number of variables in E, and equal to the number of factors in the denominator of $\frac{\alpha^\mu}{ab\ldots}$, and C is some coefficient.

A little consideration shows, that if we develope

$$\cos \alpha \left(\frac{1}{a} + \frac{1}{b} + \ldots\right)$$

in a series of powers and products of

$$\frac{1}{a}\ \frac{1}{b} \ldots\ldots\ldots\ldots,$$

and neglect all terms involving powers above the first of these quantities, the result will be $= \Sigma C \frac{\alpha^\mu}{ab\ldots}$.

Consequently

$$\Sigma C \frac{\alpha^\mu}{ab.\ldots} = 1 - \alpha^2 \Sigma \frac{1}{a}\frac{1}{b} + \alpha^4 \Sigma \frac{1}{a}\frac{1}{b}\frac{1}{c}\frac{1}{f} - \&c.$$

$\Sigma \frac{1}{a}\frac{1}{b}$ denoting the combinations two and two of the quantities $\frac{1}{a}, \frac{1}{b}\ldots$; and so of the rest. For in the developement of $(A + B + \ldots)^m$ where m is not greater than the number of quantities A, B, &c. there is always a term involving no power above the first of any of these quantities, and its coefficient is $1.2.\ldots m$. This term is obviously the sum of the combinations m and m together of A, B, &c., and that its coefficient is equal to $1.2.3.\ldots m$, appears from the polynomial theorem, viz.

$$(A + B + \ldots)^m = \Sigma \frac{1.2.\ldots m}{1.2..p.1.2\ldots q\ \&c.} A^p B^q \ldots$$

Thus $$G = \frac{p_t - \alpha^2 p_{t-2} + \alpha^4 p_{t-4} - \&c.}{mn.\ldots(a^2 + \alpha^2)(b^2 + \alpha^2).\ldots} \ldots\ldots (4);$$

where p_t, p_{t-2} &c. are the alternate coefficients of the equation

$$v^t - p_1 v^{t-1} + \&c. \quad \pm p_{t-1} v \mp p_t = 0,$$

whose roots are a, b, c &c. (I suppose t to be the number of variables in E). In precisely the same way we should find

$$H = \frac{\alpha p_{t-1} - \alpha^3 p_{t-3} + \&c.}{mn.\ldots(a^2 + \alpha^2)(b^2 + \alpha^2).\ldots} \ldots\ldots (5).$$

Next to find the values of

$$\int_0^\infty G \cos \alpha u d\alpha \text{ and } \int_0^\infty H \cos \alpha u d\alpha.$$

Let $$K = \int_0^\infty \frac{\cos \alpha u d\alpha}{(a^2 + \alpha^2)(b^2 + \alpha^2).\ldots}$$

$$= \Sigma \frac{1}{(b^2 - a^2)(c^2 - a^2).\ldots} \int_0^\infty \frac{\cos \alpha u d\alpha}{a^2 + \alpha^2};$$

$$\therefore K = \frac{\pi}{2} \Sigma \frac{1}{a(b^2 - a^2)(c^2 - a^2).\ldots} e^{\mp au} \ldots\ldots (6),$$

the upper sign is to be taken when u is > 0.

By differentiating this for a, we have the value of

$$\int_0^\infty \frac{\alpha \sin \alpha u d\alpha}{(a^2+\alpha^2)(b^2+\alpha^2).\ldots} = \pm \frac{\pi}{2} \Sigma \frac{1}{(b^2-a^2)(c^2-a^2).\ldots} e^{\mp au} \ldots (7),$$

and so by repeated differentiations we find the values of all the integrals which enter into $\int_0^\infty G \cos \alpha u d\alpha$ and $\int_0^\infty H \sin \alpha u d\alpha$.

Thus

$$F = \frac{\pi}{2}\frac{1}{mn\ldots}\,\Sigma\left\{p_t + ap_{t-1} + \&c. + a^t\right\}\frac{e^{-au}}{a(b^2-a^2)(c^2-a^2)\ldots}\ \ldots (8),$$

when u is > 0, and

$$F = \frac{\pi}{2}\frac{1}{mn\ldots}\,\Sigma\left\{p_t - ap_{t-1} + \&c. \pm a^t\right\}\frac{e^{+au}}{a(b^2-a^2)(c^2-a^2)\ldots}\ \ldots (9),$$

when u is < 0.

But $p_t - ap_{t-1} + \&c. \pm a^t = 0$.

Hence $F = 0$, when u is < 0, and therefore in (E) we may make $h_1 = 0$, so that the limits of integration for u are 0 and h.

Again $a^t + p_1a^{t-1} + \&c. = (a+a)(a+b)\ldots$

and $2a(b^2-a^2)(c^2-a^2)\ldots = \{2a.(a+b)\ldots\}\{(b-a)(c-a)\ldots\}$.

Therefore

$$\frac{a^t + p_1a^{t-1} + \&c.}{2a(b^2-a^2)(c^2-a^2)\ldots} = \frac{1}{(b-a)(c-a)\ldots}$$

and

$$F = \frac{\pi}{mn\ldots}\,\Sigma\frac{e^{-au}}{(b-a)(c-a)\ldots}\ \ldots (10),$$

and therefore finally

$$\int_0 dx \int_0 dy \ldots e^{-(max+nby+\ldots)} f(mx+ny+\ldots)$$

$$= \frac{1}{mn\ldots}\,\Sigma\frac{\int_0^h fue^{-au}du}{(b-a)(c-a)\ldots}\ \ldots\ldots (11).$$

If $a = b = c = \&c. = A$, the first side of this equation

$$= \frac{1}{mn\ldots}\frac{1}{\Gamma(t)}\int_0^h fu\,e^{-Au}u^{t-1}du \text{ by (3).}$$

As a verification of our analysis we may remark, that in this case

$$\Sigma\frac{e^{-au}}{(b-a)(c-a)\ldots} = e^{-Au}\frac{u^{t-1}}{\Gamma(t)};$$

for we have, what is probably a known result, and which at any rate may be easily proved

$$\Sigma\frac{F(a)}{(b-a)(c-a)\ldots} = \frac{1}{\Gamma(t)}F^{(t-1)}(A) \text{ when } a = b = \&c. = A,$$

where $F^{(p)}(A)$ denotes the p^{th} derived function of $F(A)$: and this formula applied to the case where $F(a) = e^{-au}$ gives the above written result.

By differentiating (11) for a, b, c, &c. λ, μ, ν, &c. times respectively, λ, μ, ν, &c. being integral or fractional, and dividing by m^λ, n^μ, p^ν, &c. we should obtain the value of

$$\int_0 dx \int_0 dy \ldots e^{-(max+nby+\ldots)} f(mx+ny+\ldots)\,x^\lambda y^\mu\ldots$$

which would include every case of (3). But the investigation would be complex, and I shall therefore only indicate it.

In a future number of the journal I may perhaps apply the method to some other cases, and particularly with regard to such multiple integrals as

$$\int dx \int dy \dots \phi(xy \dots) f[\psi(xy \dots)] F[\chi(xy \dots)], \text{ \&c.}$$

the limits being given by the series of inequalities,

$$\psi. > h_1 < h,$$
$$\chi. > k_1 < k,$$
$$\text{\&c.} > \text{\&c.} < \text{\&c.}$$

In theory, such an integral is reducible to a multiple integral of as many variables as there are limiting inequalities. But it is not easy to find cases in which this reduction can be actually effected.

II.—ON THE EQUATIONS OF MOTION OF ROTATION.

By Andrew Bell.

THE equations of the motion of rotation of a solid body, acted on by any forces, its centre of inertia being fixed, may be established in a comparatively concise manner by means of one of the theorems contained in the paper at p. 213, vol. III. of this Journal.

Let x', y', z' be the principal axes of the body passing through its centre of inertia, and let p, q, r be the angular velocities about them respectively, then these velocities are the constituents of an angular velocity ω about the instantaneous axis; also let A, B, C be the respective moments of inertia about these axes.

If x', y', z' are the constituents of the impressed forces parallel respectively to these axes, then the moments of these forces about these axes being denoted respectively by N, M, L, and Δm being an element of the body, the moment about the axis (x') is

$$N = \Sigma (y'z' - z'y') \Delta m.$$

Again, the effective forces in reference to (x') are the moving force about it arising from the constituent rotation around this axis, and also the effect of the centrifugal force arising from the rotation about the instantaneous axis in producing rotation about the axis (x'). Now A is the mass of equivalent inertia with the body, at a distance unity from the axis (x'), p is the absolute velocity at that distance, and therefore $\frac{dp}{dt}$

is the accelerating force and $A\,\dfrac{dp}{dt}$ the moving force at the same distance. Also it has been proved, in the third article of the paper referred to above, that the effect of the centrifugal force will be obtained by decomposing the rotation about the instantaneous axis into its constituent rotations about (x') and another axis (z) in the same plane with the two former, and perpendicular to (x'), and then finding the similar effect of the rotation about (z) in producing rotation about (x'). This new axis (z) is evidently the resultant axis of (y') and (z'). If, therefore ω' is the angular velocity around it and (y) an axis perpendicular to the plane of $x'y$, the required force is

$$= \omega'^2\ \Sigma yz\Delta m.$$

This expression will be transformed into one in terms of y', z' by substituting these values

$$y = y' \cos\theta - z' \sin\theta,$$
$$z = y' \sin\theta + z' \cos\theta\,;$$

whence $\omega'^2\Sigma yz\Delta m$

$$= \omega'^2 \sin\theta\cos\theta\,\Sigma(y'^2 - z'^2)\,\Delta m + \omega'^2(\cos^2\theta - \sin^2\theta)\,\Sigma y'z'\Delta m.$$

But for the principal axes $\Sigma y'z'\Delta m = 0$,

also $\quad \Sigma(y'^2 - z'^2)\,\Delta m = \Sigma\{(x'^2+y'^2)-(x'^2+z'^2)\}\Delta m = C - B\,;$

and $\quad q = \omega' \sin\theta, \quad r = \omega' \cos\theta\,;$

and therefore $\quad \omega'^2\Sigma yz\Delta m = (C - B)\,qr.$

Hence the moment of the effective forces in reference to the axis (x') is

$$= A\,\frac{dp}{dt} + (C - B)\,qr.$$

As the impressed and effective forces must equilibrate, therefore

$$A\,\frac{dp}{dt} + (C - B)\,qr - N = 0.$$

In the same manner the other two equations of the motion of rotation about the axes (y'), (z') are found to be

$$B\,\frac{dq}{dt} + (A - C)\,pr - M = 0,$$

$$C\,\frac{dr}{dt} + (B - A)\,qr - L = 0.$$

In considering the terms of any of these equations, as for instance the first, it is evident that N being the only impressed force, the other two terms must express effective forces, and the first term evidently referring to the moving force about the axis (x'), the second term must represent the

effect of centrifugal force. This has been remarked by a writer on this subject, (*Westminster Review*, vol. II. p. 321), but he has given no proof of it independently of the common demonstration. This is the only troublesome term to investigate, and to establish its origin on a new principle the theorem referred to in the first paragraph of this paper has here been employed.

III.—ON A METHOD OF FINDING THE GREATEST COMMON MEASURE OF TWO POLYNOMIALS.

By A. Q. G. CRAUFURD, M.A. Jesus College.

To find the greatest common measure of two rational and entire functions of x, X_1 and X_2.

Assume y to be one of the roots of the greatest common measure, then y must also be a root of each of the proposed polynomials X_1 and X_2, consequently it satisfies the equations

$$X_1 = 0, \quad X_2 = 0 \ldots\ldots\ldots\ldots (1).$$

The converse of this proposition is also true; viz. "whatever satisfies both these equations is a root of the greatest common measure."

To prove this, let the greatest common measure be denoted by M, and let

$$X_1 = M.Q_1, \quad X_2 = M.Q_2.$$

Then whatever is a root of X_1, must be a root of M or of Q_1; and whatever is a root of X_2 must be a root of M or of Q_2; therefore every root of X_1 *and* X_2 is a root of M or of Q_1 *and* Q_2. But Q_1 and Q_2 have no common measure, and therefore no common root; therefore, every root of X_1 *and* X_2 is a root of M. It is proved, then, that the roots common to X_1 and X_2 are roots, and the *only* roots, of M.

The problem is thus reduced to that of determining an equation which shall have all the roots which are *common* to the two equations (1), and no other roots. This is easily done by the method employed in my paper on "Elimination," in the XIIth number of this *Journal*. I will first explain the process in general terms, and then apply it to an example; and in treating the general case I shall first suppose X_1 and X_2 to be both of the n^{th} degree.

If equations (1) have all their roots in common, either of them is the equation sought, and what is required is already done. If these equations do not coincide y must have less than n values.

Take $A_n, A_{n-1}, A_{n-2} \ldots A_0$ to represent the coefficients of the powers of x in X_1, and let $B_n, B_{n-1}, B_{n-2} \ldots B_0$, be the corresponding coefficients in X_2.

Let p and q be two multipliers, such that

$$pA_n - qB_n = 0.$$

Then the equation $\quad pX_1 - qX_2 = 0$

is of the $(n-1)^{\text{th}}$ degree. Let r and s be such multipliers that

$$rA_0 - sB_0 = 0\,;$$

then the equation $\quad rX_1 - sX_2 = 0$

is of the n^{th} degree, but it is *deprived of its last term.*

It is evident that whatever satisfies equations (1) must also satisfy these. It is likewise true that whatever satisfies these must also satisfy equations (1); (except in a particular case which will be specially noticed presently). For, if we multiply the first of the two derived equations by s and the second by q, and subtract, we have

$$(ps - rq)\, X_1 = 0.$$

Again, if we multiply the first of the two derived equations by r and the second by p, and subtract, we have

$$(ps - rq)\, X_2 = 0.$$

Therefore (if $ps - rq$ is not equal to zero), the two derived equations are convertible with the two proposed. The second of the two derived equations has zero for a root, but the other has not, therefore this root is not common to the two derived nor to the two proposed; we may therefore reject it, and the two derived equations will still contain all the roots *common* to the two proposed, and they will both be of the $(n-1)^{\text{th}}$ degree.

Let $pX_1 - qX_2$ and $\dfrac{rX_1 - sX_2}{x}$ be denoted by X_1' and X_2' respectively, then $\quad X_1' = 0$ and $X_2' = 0 \ldots\ldots\ldots\ldots (2)$

are equations of the $(n-1)^{\text{th}}$ degree which are convertible with the proposed.

If these equations have all their roots in common, either of them is the equation sought. If not y can not have $(n-1)$ values. Form, therefore, from equations (2), two equations of the $(n-2)^{\text{th}}$ degree, in the same manner that equations (2) were formed from equations (1). If these coincide, and have all their roots in common, either of them is the equation sought. If not, y can not have $(n-2)$ values, and the process of reduction must be continued, and if this process leads at

last to a pair of equations which *do coincide*, and *have* all their roots in common, either of this pair of equations is the equation sought.

Let $X^p = 0$ be this equation. Then, since X^p is a function which has all the roots of the greatest common measure, and no others, it must be that common measure.

If the process of reduction never leads to a pair of equations which have their roots in common, the two proposed functions will, *by that process*, have been proved to have no common measure.

If the two proposed functions are not of the same degree, suppose X_1 to be of the m^{th} and X_2 of the n^{th} degree, m being $> n$. Assume, as before, $A_m, A_{m-1}, A_{m-2}, \ldots A_0$ to be the coefficients of the powers of x in X_1 and $B_n, B_{n-1}, B_{n-2}, \ldots B_0$ the corresponding coefficients in X_2. Let p and q be such multipliers that

$$pA_m - qB_n = 0.$$

Then the equation $$pX_1 - qx^{m-n}X_2 = 0$$

is of the $(n-1)^{\text{th}}$ degree, and this equation, together with the second of the proposed (viz. $X_2 = 0$), are convertible with the two proposed.

If we call these $$X_1' = 0, \quad X_2 = 0 \quad \ldots\ldots\ldots\ldots (2),$$

we may continue the process till we obtain two of the n^{th} degree, and then proceed as before.

It remains to notice the particular case in which the multipliers p, s, q, and r, are such that $ps - rq = 0$.

In this case we must proceed in the reduction by a different method. First form, as before, the equation

$$pX_1 - qX_2 = 0, \quad \text{or} \quad X_1' = 0.$$

This, together with $X_1 = 0$, is evidently convertible with the proposed. Let B'_{n-1}, B'_{n-2}, &c. denote the coefficients of X_1'. And let p' and q' be such that $p'A_n - q'B'_{n-1} = 0$. Then the equation $p'X_1 - q'xX'_1 = 0$ is of the $(n-1)^{\text{th}}$ degree; and the two equations $$X_1' = 0, \quad pX_1 - q'xX_1' = 0$$

are evidently convertible with

$$X_1 = 0, \text{ and } X_1' = 0;$$

therefore they are convertible with the proposed.

This latter method of reduction may always be employed, if preferred, instead of that previously given.

Ex. Let the functions be

$$2x^4 - 12x^3 + 19x^2 - 6x + 9,$$

and $$8x^3 - 36x^2 + 38x - 6.$$

Write $$2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$$
and $$4x^3 - 18x^2 + 19x - 3 = 0 \quad \text{........ (1).}$$

Multiply the first of these equations by 2, and the second by x, and subtract, and you have

$$-6x^3 + 19x^2 - 9x + 18 = 0.$$

Consequently the two equations of the third degree are

$$6x^3 - 19x^2 + 9x - 18 = 0$$
and $$4x^3 - 18x^2 + 19x - 3 = 0 \quad \text{........ (2).}$$

Multiply the first of these by 2, and the second by 3, and subtract: you obtain

$$16x^2 - 39x - 27 = 0.$$

Multiply the first of equations (2) by 1, and the second by 6, subtract and divide by x, you obtain

$$18x^2 - 89x + 105 = 0.$$

So that the two equations of the second degree are

$$16x^2 - 39x - 27 = 0$$
and $$18x^2 - 89x + 105 = 0 \quad \text{.......... (3).}$$

Multiply the first of these by 9, and the second by 8, and subtract: you obtain $$361x - 1083 = 0,$$

$$\text{or} \quad x - 3 = 0.$$

Multiply the first of equations (3) by 35, and the second by 9, add, and divide by x: you obtain

$$722x - 2166 = 0,$$

$$\text{or} \quad x - 3 = 0.$$

The two equations of the first degree *coincide*, and $x - 3$ is the greatest common measure of the proposed functions.

The process of reduction might have been performed in the following manner, which, though differing slightly in the steps, leads of course to the same result.

Take the two equations of the second degree, viz.

$$6x^3 - 19x^2 + 9x - 18 = 0$$
and $$4x^3 - 18x^2 + 19x - 3 = 0 \quad \text{........ (2).}$$

Form, as before, the equation

$$16x^2 - 39x - 27 = 0.$$

Multiply the first of equations (2) by 8, and the equation just formed by $3x$, and subtract: you obtain

$$35x^2 - 153x + 144 = 0.$$

So that the two equations of the second degree are

$$\left.\begin{array}{r} 16x^2 - 39x - 27 = 0 \\ \text{and} \quad 35x^2 - 153x + 144 = 0 \end{array}\right\} \ldots\ldots\ldots\ldots (3).$$

Multiply the first of these by 35, and the second by 16, and subtract. The result is

$$1083x - 3249 = 0,$$

$$\text{or} \quad x - 3 = 0.$$

Multiply this by $16x$, and subtract from the first of equations (3): you obtain

$$9x - 27 = 0,$$

$$\text{or} \quad x - 3 = 0.$$

As before, we find that the two equations of the first degree *coincide*, and $x - 3$ is the greatest common measure. The latter method is rather shorter than the former.

IV.—ON THE SYMMETRICAL INVESTIGATION OF POINTS OF INFLECTION.

By W. Walton, M.A. Trinity College.

The condition ordinarily adopted for the discovery of a point of inflection is, that $\dfrac{d^2y}{dx^2}$ shall change sign as x passes through the value which it has at the point; whence it follows that at the point itself $\dfrac{d^2y}{dx^2} = 0$, or $= \infty$. This method of investigation is sufficiently convenient when we have given y explicitly in a rational function of x; when, however, x and y are involved implicitly it frequently gives rise to painful, and at all times to inelegant, operations. The method which we lay before the readers of the *Journal* has the advantage of symmetry, and is, in many cases, especially when x and y are symmetrically involved in the equation to the curve, free from vexatious processes. The intelligence of the Algebraist will enable him to see in particular cases which method ought to be chosen. The symmetrical method is often very useful when the branches of a multiple point have inflection at the point.

Let the equation to an algebraical curve, cleared of radicals and negative indices, be represented by

$$F = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where F is a rational function of x and y.

Let ds denote an element of the arc of the curve at the point x, y, and let s be taken as the independent variable.

Let
$$U = \frac{dF}{dx}, \quad V = \frac{dF}{dy},$$
$$u = \frac{d^2F}{dx^2}, \quad w = \frac{d^2F}{dxdy}, \quad v = \frac{d^2F}{dy^2},$$
$$l = \frac{dx}{ds}, \quad m = \frac{dy}{ds}, \quad l' = \frac{dl}{ds}, \quad m' = \frac{dm}{ds}.$$

Then, differentiating (1), we have
$$lU + mV = 0 \ldots\ldots\ldots\ldots\ldots (2);$$
differentiating (2),
$$l^2u + 2lmw + m^2v + l'U + m'V = 0 \ldots\ldots (3).$$
Again, it is clear that $l^2 + m^2 = 1$, and therefore
$$ll' + mm' = 0 \ldots\ldots\ldots\ldots\ldots (4).$$
From (2) and (4) we get
$$l'V = m'U;$$
hence, multiplying (3) by U, we obtain
$$U(l^2u + 2lmw + m^2v) + l'(U^2 + V^2) = 0,$$
or, by virtue of (2),
$$\frac{l^2U}{V^2}(uV^2 - 2UVw + U^2v) + l'(U^2 + V^2) = 0 \ldots (5).$$
In a similar way we may shew that
$$\frac{m^2V}{U^2}(uV^2 - 2UVw + U^2v) + m'(U^2 + V^2) = 0 \ldots (6).$$
Again, by the relation $l^2 + m^2 = 1$ and the equation (2), we get
$$\frac{l^2}{V^2} = \frac{1}{U^2 + V^2} = \frac{m^2}{U^2}:$$
hence (5) and (6) give us
$$U(uV^2 - 2UVw + U^2v) + l'(U^2 + V^2)^2 = 0 \ldots . (7),$$
$$V(uV^2 - 2UVw + U^2v) + m'(U^2 + V^2)^2 = 0 \ldots . (8).$$
Now at a point of inflection it is evident that l and m must be the one a maximum and the other a minimum: hence, as we pass in the neighbourhood of the point along the curve from one side of the point to the other, we know by the theory of maxima and minima that l' and m' must each of them change sign. It is evident, then, from (7) and (8), that
$$U(uV^2 - 2UVw + U^2v)$$

and $$V(uV^2 - 2UVw + U^2v)$$
must both change sign.

Suppose first that neither U nor V changes sign as we pass through the point; then the sufficient and necessary condition for a change of sign in the value of l' and m', is that

$$uV^2 - 2UVw + U^2v \ldots\ldots\ldots\ldots (9)$$

change sign as we pass through the point; this condition evidently involves the fact, that at the point itself

$$uV^2 - 2UVw + U^2v = 0 \ldots\ldots\ldots\ldots (10).$$

Next suppose that U changes sign; then it is evident that the expression (9) must not change sign, for otherwise l', as will be evident from the formula (7), could not change sign. But from (8) we see that for a change of sign in the value of m', one and one only of the quantities V and (9) must change sign; hence V only must change sign. Thus we see that if either of the quantities U and V change sign, both must do so, and that (9) must not change sign. If U and V both change sign it is clear that at the point itself

$$U = 0, \quad V = 0,$$

which are the conditions for multiplicity of branches at the point.

The general rule, therefore, for finding points of inflection may be thus enunciated. First ascertain the values of x and y which will satisfy simultaneously the equations

$$F = 0, \quad uV^2 - 2UVw + vU^2 = 0,$$

and reject all the pairs of values thus obtained which do not, as we pass through the corresponding points, correspond to a change of sign in the expression

$$uV^2 - 2UVw + vU^2,$$

or which do correspond to a change of sign in either U or V; the pairs of values of x and y, which are retained, correspond to points of inflection.

Secondly, ascertain those pairs of values which satisfy simultaneously $$F = 0, \quad U = 0, \quad V = 0,$$

and reject all of these pairs which do not correspond to a change of sign in both U and V as we pass through the corresponding points along one or other of the branches, or which do correspond to a change of sign in the expression

$$uV^2 - 2UVw + vU^2.$$

In the preceding investigation we have supposed F to be a rational function of x and y. Should this not be the case it

will be evident, from what has been said, that in addition to the values of x and y, which may be obtained by the rule which we have enunciated, we must likewise take those which will render in the first case,

$$F = 0, \quad uV^2 - 2UVw + vU^2 = \infty;$$

and in the second case,

$$F = 0, \quad U = \infty, \quad V = \infty;$$

the conditions depending on change of sign being the same as before.

Ex. 1. Let the curve be

$$F = ax^3 + by^3 - c^4 = 0.$$

Then
$$U = 3ax^2, \quad V = 3by^2,$$
$$u = 6ax, \quad w = 0, \quad v = 6by.$$

Hence, from the formula (10) there is, if we cast out constant factors,

$$xy\,(ax^3 + by^3) = 0;$$

or, by the equation to the curve,

$$xy = 0.$$

Thus $x = 0$, or $y = 0$, and in both cases neither U nor V changes sign, while the formula (9) does change sign. Hence we have two points of inflection

$$x = 0, \quad y = \frac{c^{\frac{4}{3}}}{b^{\frac{1}{3}}},$$

and
$$x = \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}, \quad y = 0.$$

Ex. 2. Take the curve

$$F = (x^2 + y^2)^2 - a^2x^2 + b^2y^2,$$

and suppose that we wish to find whether there be a point of inflection at the origin. Then

$$U = 2x\,(2x^2 + 2y^2 - a^2),$$
$$V = 2y\,(2x^2 + 2y^2 + b^2),$$
$$u = 12x^2 + 4y^2 - 2a^2,$$
$$v = 4x^2 + 12y^2 + 2b^2,$$
$$w = 8xy.$$

From these results it is evident that U and V both change sign if we change x and y each of them from ± 0 to ∓ 0.

Moreover it is clear that neither uV^2, vU^2, nor $2UVw$, experience any change of sign when we put $\pm x$, $\pm y$ for $\mp x$, $\mp y$ respectively. Hence the expression (9) does not change sign. If we had kept y positive or negative throughout while we changed x from ± 0 to ∓ 0, the expression (9) would have changed sign, and flexure would not have taken place. Hence we see that the branch which passes through the origin from below to above the axis of x, or that which passes from above to below, will have an inflection at the origin. The discussion of this example by the unsymmetrical method would have been much more embarrassing.

Ex. 3. Let the curve be

$$F = \left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}} - 1 = 0.$$

Then
$$U = \frac{1}{3a}\left(\frac{x}{a}\right)^{-\frac{2}{3}}, \quad V = \frac{1}{3b}\left(\frac{y}{b}\right)^{-\frac{2}{3}},$$

$$u = -\frac{2}{9a^2}\left(\frac{x}{a}\right)^{-\frac{5}{3}}, \quad w = 0, \quad v = -\frac{2}{9b^2}\left(\frac{y}{b}\right)^{-\frac{5}{3}}.$$

Hence the expression

$$uV^2 - 2UVw + vU^2$$

will vary as
$$\left(\frac{x}{a}\right)^{-\frac{5}{3}}\left(\frac{y}{b}\right)^{-\frac{4}{3}} + \left(\frac{y}{b}\right)^{-\frac{5}{3}}\left(\frac{x}{a}\right)^{-\frac{4}{3}}$$

$$= \left(\frac{x}{a}\right)^{-\frac{5}{3}}\left(\frac{y}{b}\right)^{-\frac{5}{3}}\left\{\left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}}\right\}$$

$$= \left(\frac{x}{a}\right)^{-\frac{5}{3}}\left(\frac{y}{b}\right)^{-\frac{5}{3}}.$$

Putting this expression $= \infty$, we get $x = 0$, or $y = 0$; and therefore by the equation to the curve $y = b$, $x = a$, respectively. It is evident then that as x passes through 0, U and V do not change sign while the expression (9) does: and similarly for y; hence there are two points of inflection, viz. $x = 0$, $y = b$, and $x = a$, $y = 0$.

V.—DEMONSTRATION OF PASCAL'S THEOREM.

By A. Cayley, B.A. Fellow of Trinity College.

Lemma 1. Let $U = Ax + By + Cz = 0$ be the equation to a plane passing through a given point taken for the origin, and consider the planes

$$U_1 = 0,\quad U_2 = 0,\quad U_3 = 0,\quad U_4 = 0,\quad U_5 = 0,\quad U_6 = 0.$$

The condition which expresses that the intersections of the planes (1) and (2), (3) and (4), (5) and (6) lie in the same plane, may be written down under the form

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 & . & . \\ B_1 & B_2 & B_3 & B_4 & . & . \\ C_1 & C_2 & C_3 & C_4 & . & . \\ . & . & A_3 & A_4 & A_5 & A_6 \\ . & . & B_3 & B_4 & B_5 & B_6 \\ . & . & C_3 & C_4 & C_5 & C_6 \end{vmatrix} = 0.$$

Lemma 2. Representing the determinants

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \text{ \&c.}$$

by the abbreviated notation $\overline{123}$, &c. The following equation is identically true:

$$\overline{345}\,.\,\overline{126} - \overline{346}\,.\,\overline{125} + \overline{356}\,.\,\overline{124} - \overline{456}\,.\,\overline{123} = 0.$$

This is an immediate consequence of the equations

$$\begin{vmatrix} . & . & x_3 & x_4 & x_5 & x_6 \\ . & . & y_3 & y_4 & y_5 & y_6 \\ . & . & z_3 & z_4 & z_5 & z_6 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{vmatrix} = \begin{vmatrix} . & . & x_3 & x_4 & x_5 & x_6 \\ . & . & y_3 & y_4 & y_5 & y_6 \\ . & . & z_3 & z_4 & z_5 & z_6 \\ x_1 & x_2 & . & . & . & . \\ y_1 & y_2 & . & . & . & . \\ z_1 & z_2 & . & . & . & . \end{vmatrix} = 0.$$

Consider now the points 1, 2, 3, 4, 5, 6, the co-ordinates of these being respectively $x_1, y_1, z_1 \ldots x_6, y_6, z_6$. I represent, for shortness, the equation to the plane passing through the origin, and the points 1, 2, which may be called the plane $\overline{12}$, in the form

$$x\,.\,\overline{12}_x + y\,.\,\overline{12}_y + z\,.\,\overline{12}_z = 0.$$

Consequently the symbols $\overline{12}_x$, $\overline{12}_y$, $\overline{12}_z$ denote respectively $y_1z_2 - y_2z_1$, $z_1x_2 - z_2x_1$, $x_1y_2 - x_2y_1$, and similarly for the planes $\overline{13}$, &c. If now the intersections of $\overline{12}$ and $\overline{45}$, $\overline{23}$ and $\overline{56}$,

$\overline{34}$ and $\overline{61}$ lie in the same plane, we must have, by Lemma (1), the equation

$$\begin{vmatrix} 12_x & 45_x & 23_x & 56_x & . & . \\ 12_y & 45_y & 23_y & 56_y & . & . \\ 12_z & 45_z & 23_z & 56_z & . & . \\ . & . & 23_x & 56_x & 34_x & 61_x \\ . & . & 23_y & 56_y & 34_y & 61_y \\ . & . & 23_z & 56_z & 34_z & 61_z \end{vmatrix} = 0.$$

Multiplying the two sides of this equation by the two sides respectively of the equation

$$\begin{vmatrix} x_6 & x_1 & x_2 & . & . & . \\ y_6 & y_1 & y_2 & . & . & . \\ z_6 & z_1 & z_2 & . & . & . \\ . & . & . & x_3 & x_4 & x_5 \\ . & . & . & y_3 & y_4 & y_5 \\ . & . & . & z_3 & z_4 & z_5 \end{vmatrix} = \overline{612} \,.\, \overline{345}.$$

And observing the equations

$$x_6\overline{12}_x + y_6\overline{12}_y + z_6\overline{12}_z = \overline{612}, \quad \overline{112} = 0, \text{ \&c.}$$

This becomes

$$\begin{vmatrix} 612 & . & . & . & . & . \\ 645 & 145 & 245 & . & . & . \\ 623 & 123 & . & . & 423 & 523 \\ . & 156 & 256 & 356 & 456 & . \\ . & . & . & 361 & 461 & 561 \end{vmatrix} = 0.$$

Reducible to

$$\overline{612}\ \overline{534} \begin{vmatrix} \overline{145} & \overline{245} & . & . \\ 123 & . & . & 423 \\ 156 & 256 & 356 & 456 \\ . & . & 361 & 461 \end{vmatrix} = 0.$$

Or, omitting the factor $\overline{612}\ \overline{534}$ and expanding,

$$\overline{145}\,.\,\overline{256}\,.\,\overline{423}\,.\,\overline{361} + \overline{245}\,.\,\overline{123}\,.\,\overline{456}\,.\,\overline{361}$$
$$- \overline{245}\,.\,\overline{123}\,.\,\overline{356}\,.\,\overline{461} - \overline{245}\,.\,\overline{156}\,.\,\overline{423}\,.\,\overline{361} = 0.$$

Considering for instance x_6, y_6, z_6 as variable, this equation expresses evidently that the point (6) lies in a cone of the second order having the origin for its vertex, and the equation is evidently satisfied by writing $x_6, y_6, z_6 = x_1, y_1, z_1$ or x_3, y_3, z_3 or x_4, y_4, z_4 or x_5, y_5, z_5, or the cone passes through the

points 1, 3, 4, 5. For $x_6, y_6, z_6 = x_2, y_2, z_2$, the equation becomes, reducing and dividing by $\overline{245}\ \overline{123}$,

$$\overline{452}\,.\,\overline{321} - \overline{352}\,.\,\overline{421} + \overline{152}\,.\,\overline{423} = 0,$$

which is deducible from Lemma (2), by writing $x_6, y_6, z_6 = x_2, y_2, z_2$, and is therefore identically true. Hence the cone passes through the point (2), and therefore the points 1, 2, 3, 4, 5, 6 lie in the same cone of the second order, which is Pascal's Theorem. I have demonstrated it in the cone, for the sake of symmetry; but by writing throughout unity instead of z, the above applies directly to the case of the theorem in the plane.

The demonstration of Chasles' form of Pascal's Theorem (viz. that the anharmonic relation of the planes $\overline{61}, \overline{62}, \overline{63}, \overline{64}$ is the same with that of $\overline{51}, \overline{52}, \overline{53}, \overline{54}$), is very much simpler; but as it would require some preparatory information with reference to the analytical definition of the similarity of anharmonic relation, I must defer it to another opportunity.

VI.—ON THE TRANSFORMATION OF MULTIPLE INTEGRALS.

By George Boole.

The result of the elimination of the variables from the equations

$$\left.\begin{array}{l} a_1x_1 + a_2x_2.\,.\,.\,.+ a_nx_n = 0 \\ b_1x_1 + b_2x_2.\,.\,.\,.+ b_nx_n = 0 \\ \dots\dots\dots\dots\dots\dots \\ r_1x_1 + r_2x_2.\,.\,.\,.+ r_nx_n = 0 \end{array}\right\} \quad \dots\dots\dots (1)$$

is an equation of which the second member is 0, and of which the first member is formed from the coefficient of $x_1x_2..x_n$ in the product of the given equations, by assuming a particular term, as $a_1b_2..r_n$, positive, and applying to every other term a change of sign, for every binary permutation of the suffixes which it may exhibit, when compared with the proposed term, $a_1b_2..r_n$.

By a binary permutation is here meant a permutation affecting only the relative position of two quantities, as 123, 132; or 123, 213.

The truth of the above theorem is shewn by the following considerations. The elimination of x_1 from the first and second equation of the system (1) introduces terms of the form $a_1b_2 - a_2b_1$, $a_1b_3 - a_3b_1$, &c., in which the law of binary permutation is apparent, and as we may begin the process of elimination with any variable and with any pair of equations,

the law is universal. From the same instance it is evident that no proposed suffix can occur twice in a given term, which condition is also characteristic of the coefficient of $x_1 x_2 . . x_n$ in the product of the equations of the system, whence the theorem is manifest.

As an example let $n = 4$, and assuming the term $a_1 b_2 c_3 d_4$ positive, let it be required to find the signs of the terms $a_1 b_3 c_2 d_4$, $a_4 b_2 c_1 d_3$, $a_3 b_4 c_2 d_1$. We proceed thus:

1324	4213	3421
1234	1243	1423
	1234	1243
		1234

Here the term $a_1 b_3 c_2 d_4$ is reduced to $a_1 b_2 c_3 d_4$ by a single binary permutation of the suffixes, viz. 32 to 23, wherefore the sign is negative; the term $a_4 b_2 c_1 d_3$ undergoing two permutations is positive, and the term $a_3 b_4 c_2 d_1$ undergoing three is negative.

The above theorem may be conveniently applied when we wish to ascertain the result of elimination from a system of equations, to which, from their number, it might be difficult to apply a general form. If, for example, we had the system

$$\left.\begin{array}{r} a_1 x_1 = 0 \\ b_1 x_1 + b_2 x_2 = 0 \\ c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ r_1 x_1 + r_2 x_2 . . + r_n x_n = 0 \end{array}\right\} \cdots\cdots\cdots\cdots (2),$$

our theorem would at once give

$$a_1 b_2 c_3 . . r_n = 0,$$

a result to which we shall have occasion to refer.

In what follows we shall employ the term "final derivative" to the first member of the equation expressing the final result of elimination from a proposed system of equations.

Now let it be required to transform the multiple integral $\int\int . . dx_1 dx_2 . . dx_n$ into one depending on the variables $u_1 u_2 . . u_n$, by virtue of the equations $x_1 = U_1$, $x_2 = U_2$, $x_n = U_n$. Differentiating, we have

$$\left.\begin{array}{l} dx_1 = \dfrac{dU_1}{du_1} du_1 + \dfrac{dU_1}{du_2} du_2 . . + \dfrac{dU_1}{du_n} du_n \\ dx_2 = \dfrac{dU_2}{du_1} du_1 + \dfrac{dU_2}{du_2} du_2 . . + \dfrac{dU_2}{du_n} du_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ dx_n = \dfrac{dU_n}{du_1} du_1 + \dfrac{dU_n}{du_2} du_2 . . + \dfrac{dU_n}{du_n} du_n \end{array}\right\} \cdots (3).$$

Now when x_1 varies, $x_2, x_3 . . x_n$ are constant, and $dx_2 . . dx_n$ vanish. Hence, if for simplicity we write $\frac{dU_1}{du_1} = a_1$, $\frac{dU_2}{du_1} = b_1$, &c., we have, on the above condition,

$$\left.\begin{array}{l} dx_1 = a_1du_1 + a_2du_2 . . + a_ndu_n \\ 0 = b_1du_1 + b_2du_2 . . + b_ndu_n \\ . . . . \\ 0 = r_1du_1 + r_2du_2 . . + r_ndu_n \end{array}\right\} \ldots\ldots (4),$$

whence a relation may be found connecting x_1 and u_1. Let $v_1 = 1$, then the above system of equations may be thus written:

$$\left.\begin{array}{r} (a_1du_1 - dx_1)\, v_1 + a_2du_2 . . + a_ndu_n = 0 \\ (b_1du_1)\, v_1 + b_2du_2 . . + b_ndu_n = 0 \\ . . . . \\ (r_1du_1)\, v_1 + r_2du_2 . . + r_ndu_n = 0 \end{array}\right\} \ldots . (5).$$

The result of the elimination of $v_1, du_2 . . du_n$ from these equations, being linear with respect to the coefficients of v_1, will be of the form

$$Ldx_1 + Mdu_1 = 0 \ldots\ldots\ldots\ldots (6).$$

In the system (5) let $dx_1 = 0$, then the result of elimination is evidently

$$E_1du_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

where E_1 is the final derivative of the second members of (4) equated to 0, whence $M = E_1$. Again in (5) let $du_1 = 0$, we have

$$\begin{array}{r} (-dx)\, v_1 + a_2du_2 . . + a_ndu_n = 0, \\ b_2du_2 . . + b_ndu_n = 0, \\ . . . . \\ r_2du_2 . . + r_ndu_n = 0\,; \end{array}$$

whence by the theorem, as the result of elimination,

$$-E_2dx = 0,$$

where E_2 is the final derivative of the system

$$\begin{array}{r} b_2du_2 + b_3du_3 . . + b_ndu_n = 0, \\ c_2du_2 + c_3du_3 . . + c_ndu_n = 0, \\ . . . . \\ r_2du_2 + r_3du_3 . . + r_ndu_n = 0\,; \end{array}$$

wherefore $L = -E_2$, and

$$-E_2dx_1 + E_1du_1 = 0,$$

$$dx_1 = \frac{E_1}{E_2}\, du_1 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now, resuming (3), let x_2 vary, then $dx_1 dx_3 . . dx_n$ vanish, wherefore by (8), $du_1 = 0$, and we have

$$\left.\begin{array}{r} dx_2 = b_2 du_2 + b_3 du_3 . . + b_n du_n \\ 0 = c_2 du_2 + c_3 du_3 . . + c_n du_n \\ 0 = r_2 du_2 + r_3 du_3 . . + r_n du_n \end{array}\right\} \text{.......(9).}$$

Proceeding with this system of equations as with (4), we get

$$dx_2 = \frac{E_2}{E_3} du_2 \text{ ....(10),}$$

where E_3 is the final derivative of the system

$$c_3 du_3 + c_4 du_4 . . + c_n du_n = 0,$$

$$\text{....................}$$

$$r_3 du_3 + r_4 du_4 . . + r_n du_n = 0\ ;$$

finally we have $\qquad dx_n = E_n du_n$ ....(11);

wherein $E_n = r_n$. Now multiply (8), (10), . . (11) together, and there results

$$dx_1 dx_2 . . dx_n = E_1 du_1 du_2 . . du_n \text{.......(12).}$$

The sign of E_1 is ambiguous, for it will depend on the order in which $du_1, du_2, . . du_n$ are eliminated,—a thing in its nature indifferent.

Now to extend this theorem to the case in which the two sets of variables are indiscriminately mixed, let $V_1 = 0$, $V_2 = 0$, $V_n = 0$ be the proposed equations. Differentiating we have

$$\left.\begin{array}{l} \dfrac{dV_1}{dx_1} dx_1 . . + \dfrac{dV_1}{dx_n} dx_n = - \dfrac{dV_1}{du_1} du_1 . . - \dfrac{dV_1}{du_n} du_n \\ \dfrac{dV_n}{dx_1} dx_1 . . + \dfrac{dV_n}{dx_n} dx_n = - \dfrac{dV_n}{du_1} du_1 . . - \dfrac{dV_n}{du_n} du_n \end{array}\right\}$$

Let E be the final derivative of the first members equated to 0, E' that of the second members equated to 0, then $\dfrac{E'}{E} = E_1$, by the theory of linear transformations (*Journal*, vol. III. p. 109), whence

$$dx_1 dx_2 . . dx_n = \frac{E'}{E} du_1 du_2 . . du_n \text{ ...... (13).}$$

On account of the ambiguous signs of E and E', it is not even necessary that the terms involving $du_1\, du_2 . . du_n$ should be transposed to the second members. The general result of our investigations may therefore be expressed in the following Rule.

To transform any multiple integral, $\int\int..X dx_1 dx_2..dx_n$, into one depending on $u_1, u_2, ..u_n$, having given the equations connecting the two sets of variables.

RULE. *Differentiate the given equations relatively to $x_1, x_2..x_n$, and eliminating $dx_1dx_2..dx_n$, let E be the final derivative. Proceed in the same manner with $u_1, u_2..u_n$, and let E' be the corresponding final derivative, then*

$$dx_1dx_2..dx_n = \frac{E'}{E} du_1du_2..du_n. \ldots \ldots (14).$$

Ex. 1. Let the equations of transformation be

$$x_1 = r \cos \theta_1,$$
$$x_2 = r \sin \theta_1 \cos \theta_2,$$
$$x_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3,$$
$$x_{n-1} = r \sin \theta_1 \sin \theta_2..\sin \theta_{n-2} \cos \theta_{n-1},$$
$$x_n = r \sin \theta_1 \sin \theta_2..\sin \theta_{n-1}.$$

Squaring and adding, we have

$$x_1^2 + x_2^2.. + x_n^2 = r^2,$$

which we shall employ in place of the last equation of the system. Now, differentiating with respect to $x_1x_2..x_n$, we get

$$dx_1 = 0,$$
$$dx_2 = 0,$$
$$2x_1dx_1.. + 2x_ndx_n = 0;$$

whence, by (2), $E = 2x_n$. Again, differentiating with respect to $r, \theta_1,..\theta_{n-1}$, we have

$$2rdr = 0,$$
$$\cos \theta_1 dr - r \sin \theta_1 d\theta_1 = 0,$$
$$.. - r \sin \theta_1 \sin \theta_2.. \sin \theta_{n-1}\, d\theta_{n-1} = 0;$$

whence $E' = (-)^{n-1}\, 2r^n (\sin \theta_1)^{n-1} (\sin \theta_2)^{n-2}.. \sin \theta_{n-1}$, and

$$dx_1dx_2..dx_n = \pm \frac{r^n (\sin \theta_1)^{n-1} (\sin \theta_2)^{n-2}.. \sin \theta_{n-1}}{x_n} drd\theta_1..d\theta_{n-1}$$
$$= \pm r^{n-1} (\sin \theta_1)^{n-2} (\sin \theta_2)^{n-3}.. \sin \theta_{n-2} d\theta_1 d\theta_2..d\theta_{n-1}.$$

If $n = 3$, the above is the transformation from rectangular to polar co-ordinates.

Ex. 2. Let the equations of transformation be

$$x_1 = 1 - u_1,$$
$$x_2 = u_1(1 - u_2),$$
$$x_3 = u_1u_2(1 - u_3),$$
$$x_n = u_1u_2..u_{n-1}(1 - u_n).$$

Here $E = 0$, and differentiating with respect to $u_1, u_2, \ldots u_n$,

$$\begin{aligned} -du_1 &= 0, \\ (1-u_2)\,du_1 - u_1du_2 &= 0, \\ \ldots\ldots - u_1u_2du_3 &= 0, \\ \ldots\ldots - u_1u_2\ldots u_{n-1}\,du_n &= 0\,; \end{aligned}$$

whence $E' = (-)^n u_1^{n-1}u_2^{n-2}\ldots u_{n-1}$, wherefore

$$dx_1dx_2\ldots dx_n = \pm\, u_1^{n-1}u_2^{n-2}\ldots u_{n-1}du_1du_2\ldots du_n \ldots (15).$$

The above transformation leads to a very elegant proof of M. Liouville's theorem expressing the value of the definite multiple integral

$$\iint\ldots dx_1dx_2\ldots dx_n x_1^{\alpha-1}x_2^{\beta-1}\ldots x_n^{\gamma-1}f(x_1+x_2\ldots+x_n)\ldots(16),$$

the limits of the variables being defined by the inequality

$$x_1+x_2\ldots+x_n \leqq 1.$$

We find $x_1+x_2 = 1-u_1u_2$, $x_1+x_2+x_3 = 1-u_1u_2u_3$; and, by induction, $x_1+x_2\ldots+x_n = 1-u_1u_2\ldots u_n$. This, with the preceding, reduces the integral to the form

$$\iint\ldots du_1du_2\ldots u_1^{\beta+\gamma\ldots}u_2^{\gamma+\delta\ldots}(1-u_1)^{\alpha-1}(1-u_2)^{\beta-1}\ldots f(1-u_1u_2\ldots u_n)\ldots(17),$$

the limits of each variable being 0 and 1.

Now, in general, $f(t) = t^{\frac{d}{d\theta}}f(\varepsilon^\theta)$, if $\theta = 0$. For let $t = \varepsilon^\phi$, then $f(t) = f(\varepsilon^{\phi+\theta}) = \varepsilon^{\phi\frac{d}{d\theta}}f(\varepsilon^\theta)$, by Taylor's theorem, $= t^{\frac{d}{d\theta}}f(\varepsilon^\theta)$. Hence $f(1-u_1u_2\ldots u_n) = (u_1u_2\ldots u_n)^{\frac{d}{d\theta}}f(1-\varepsilon^\theta)$. Substituting this expression in (17), we get

$$\iint\ldots du_1du_2\ldots u_1^{\frac{d}{d\theta}+\beta+\gamma\ldots}u_2^{\frac{d}{d\theta}+\gamma+\delta\ldots}\ldots(1-u_1)^{\alpha-1}(1-u_2)^{\beta-1}\ldots f(1-\varepsilon^\theta)\ldots(18).$$

But, by the well-known relation connecting the first and the second of the Eulerian integrals,

$$\int_0^1 du_1u_1^{\frac{d}{d\theta}+\beta+\gamma\ldots}(1-u_1)^{\alpha-1} = \frac{\Gamma\left(\frac{d}{d\theta}+1+\beta+\gamma\ldots\right)\Gamma(\alpha)}{\Gamma\left(\frac{d}{d\theta}+1+\alpha+\beta\ldots\right)},$$

$$\int_0^1 du_2u_2^{\frac{d}{d\theta}+\gamma+\delta\ldots}(1-u_2)^{\beta-1} = \frac{\Gamma\left(\frac{d}{d\theta}+1+\gamma\ldots\right)\Gamma(\beta)}{\Gamma\left(\frac{d}{d\theta}+1+\beta\ldots\right)};$$

and finally

$$\int_0^1 du_n u_n^{\frac{d}{d\theta}}(1-u_n)^{\nu-1} = \frac{\Gamma\left(\frac{d}{d\theta}+1\right)\Gamma(\nu)}{\Gamma\left(\frac{d}{d\theta}+1+\nu\right)}.$$

Substituting these values in (18), we have simply

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)..\Gamma(\nu)\,\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}+1+\alpha+\beta..\right)} f(1-\varepsilon^\theta)$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)..\Gamma(\nu)}{\Gamma(\alpha+\beta..+\nu)} \; \frac{\Gamma(\alpha+\beta..+\nu)\,\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}+1+\alpha+\beta..\right)} f(1-\varepsilon^\theta)$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)..\Gamma(\nu)}{\Gamma(\alpha+\beta..+\nu} \int_0^1 dv\, v^{\alpha+\beta...+\nu-1}(1-v)^{\frac{d}{d\theta}} f(1-\varepsilon^\theta).$$

Now $(1-v)^{\frac{d}{d\theta}} f(1-\varepsilon^\theta) = f\{1-(1-v)\} = f(v)$, whence the expression becomes

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)..\Gamma(\nu)}{\Gamma(\alpha+\beta..+\nu)} \int_0^1 dv\, v^{\alpha+\beta...+\nu-1} f(v).$$

As a final example I propose to consider the definite multiple integral

$$V = \iint.. \frac{dx_1 dx_2..dx_{n-1}}{x^n} \{f(a_1x_1..+a_{n-1}x_{n-1}+a_nx_n) + f(a_1x_1..+a_{n-1}x_{n-1}-a_nx_n)\},$$

the integrations extending to all real values of $x_1x_2...x_{n-1}$, and to all real and positive values of x_n which satisfy the conditions

$$x_1^2+x_2^2...+x_{n-1}^2 \overline{\overline{<}}\, 1, \qquad x_1^2+x_2^2...+x_n^2 = 1.$$

The value of this definite integral is

$$V = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi d\theta \sin\theta^{n-2}\cos\theta\, f(A\cos\theta)....(20),$$

wherein $A = (a_1^2+a_2^2...+a_n^2)^{\frac{1}{2}}$, and it is found by integrating with respect to a_n in the general formula which I have given in a paper entitled, *Remarks on a Theorem of M. Catalan*, (*Journal*, vol. III. p. 281).

Let us assume $x_1 = \cos\theta_1$,

$$x_2 = \sin\theta_1\cos\theta_2,$$
$$x_{n-1} = \sin\theta_1 ... \sin\theta_{n-2}\cos\theta_{n-1},$$
$$x_n = \sin\theta_1\sin\theta_2 ... \sin\theta_{n-1}.$$

Now while $x_1, x_2 \ldots x_{n-1}$ vary from -1 to 1, the dependent variable x_n is restricted to positive values. These conditions are satisfied by taking 0 and π for the limits of each of the new variables $\theta_1, \theta_2 \ldots \theta_{n-1}$. Considering the $n-1$ first equations of the preceding system, we have $E=0$, and differentiating with respect to $\theta_1, \theta_2 \ldots \theta_{n-1}$,

$$-\sin\theta_1 d\theta_1 = 0,$$
$$\cos\theta_1 \cos\theta_2 d\theta_1 - \sin\theta_1 \sin\theta_2 d\theta_2 = 0,$$
$$\ldots\ldots - \sin\theta_1 \ldots \sin\theta_{n-1}\, d\theta_{n-1} = 0;$$

whence $E' = \sin\theta_1^{\,n-1} \sin\theta_2^{\,n-2} \ldots \sin\theta_{n-1}$, wherefore

$$\frac{dx_1 dx_2 \ldots dx_{n-1}}{x_n} = \frac{\sin\theta_1^{\,n-1} \ldots \sin\theta_{n-1}}{\sin\theta_1 \ldots \sin\theta_{n-1}}\, d\theta_1 d\theta_2 \ldots d\theta_{n-1}$$
$$= \sin\theta_1^{\,n-2} \sin\theta_2^{\,n-3} \ldots \sin\theta_{n-2}\, d\theta_1 d\theta_2 \ldots d\theta_{n-1},$$

and the transformed integral may be thus written:

$$\int_0^\pi \int_0^\pi \ldots d\theta_1 \ldots d\theta_{n-1} \sin\theta_1^{\,n-2} \ldots \sin\theta_{n-2} f(a_1 \cos\theta_1 \ldots$$
$$+ a_{n-1} \sin\theta_1 \ldots \sin\theta_{n-2} \cos\theta_{n-1} + a_n \sin\theta_1 \ldots \sin\theta_{n-1})$$
$$+ \int_0^\pi \int_0^\pi \ldots d\theta_1 \ldots d\theta_{n-1} \sin\theta_1^{\,n-2} \ldots \sin\theta_{n-2} f(a_1 \cos\theta_1 \ldots$$
$$+ a_{n-1} \sin\theta_1 \ldots \sin\theta_{n-2} \cos\theta_{n-1} - a_n \sin\theta_1 \ldots \sin\theta_{n-1}) \ldots (20).$$

In the second term of the above expression let $\theta_{n-1} = 2\pi - \theta'_{n-1}$, then $d\theta_{n-1} = -d\theta'_{n-1}$, $\cos\theta'_{n-1} = \cos\theta'_{n-1}$, $\sin\theta_{n-1} = -\sin\theta'_{n-1}$, also the limits of θ'_{n-1} are 2π and π, wherefore the proposed term becomes

$$-\int_0^\pi \int_0^\pi \ldots \int_{2\pi}^{\pi} d\theta_1 \ldots d\theta'_{n-1} \sin\theta_1^{\,n-2} \ldots \sin\theta_{n-2}$$
$$f(a_1 \cos\theta_1 \ldots + a_{n-1} \sin\theta_1 \ldots \cos\theta'_{n-1} + a_n \sin\theta_1 \ldots \sin\theta'_{n-1})$$
$$= \int_0^\pi \int_0^\pi \ldots \int_\pi^{2\pi} d\theta_1 \ldots d\theta'_{n-1} \sin\theta_1^{\,n-2} \ldots \sin\theta_{n-2}$$
$$f(a_1 \cos\theta_1 \ldots + a_{n-1} \sin\theta_1 \ldots \cos\theta'_{n-1} + a_n \sin\theta_1 \ldots \sin\theta'_{n-1}).$$

Taking away the accent from θ'_{n-1} in the above term, and annexing the result to the first term of (20), we have

$$\int_0^\pi \int_0^\pi \ldots \int_0^{2\pi} d\theta_1 \ldots d\theta_{n-1} \sin\theta_1^{\,n-2} \ldots \sin\theta_{n-2}$$
$$f(a_1 \cos\theta_1 \ldots + a_{n-1} \sin\theta_1 \ldots \cos\theta_{n-1} + a_n \sin\theta_1 \sin\theta_{n-1})$$
$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi d\theta \sin\theta^{n-2} \cos\theta\, f(A\cos\theta) \ldots (21),$$

which is the transformation contemplated.

Some remarkable deductions from the general theorem (20) may properly be noticed here. Let $a_n = 0$, then in the result, writing n for $n-1$, and for x_{n+1} its value $\sqrt{(1 - x_1^2 - x_2^2 \dots x_n^2)}$, we have

$$\iint \dots dx_1 dx_2 \dots dx_n \frac{f(a_1 x_1 + a_2 x_2 \dots a_n x_n)}{\sqrt{(1 - x_1^2 \dots - x_n^2)}}$$

$$= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\pi d\theta \sin\theta^{n-1} \cos\theta\, f(A\cos\theta) \dots (22),$$

the integrations in the first member extending to all real values $x_1 x_2 \dots x_n$ subject to the condition

$$x_1^2 + x_2^2 \dots x_n^2 \overline{\overline{<}}\, 1 \dots\dots\dots\dots (23).$$

Performing on both sides of the equation the operation $\left(\frac{d}{da_1}\right)^\alpha \left(\frac{d}{da_2}\right)^\beta \dots \left(\frac{d}{da_n}\right)^\nu$, we arrive at a result which may be thus expressed:

$$\iint \dots dx_1 dx_2 \dots dx_n x_1^\alpha x_2^\beta \dots x_n^\nu \frac{F(a_1 x_1 + a_2 x_2 \dots a_n x_n)}{\sqrt{(1 - x_1^2 \dots - x_n^2)}} \dots\dots (24)$$

$$= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{d}{da_1}\right)^\alpha \left(\frac{d}{da_2}\right)^\beta \dots \left(\frac{d}{da_n}\right)^\nu \int_0^\pi d\theta \sin\theta^{n-1} \cos\theta\, f(A\cos\theta),$$

wherein

$f(t) = \iint \dots F(t)\, dt^{\alpha+\beta\dots+\nu}$, $A = (a_1^2 + a_2^2 \dots + a_n^2)^{\frac{1}{2}}$, and $x_1, x_2, \dots x_n$ are subject to the condition (23). The theorem is therefore applicable to every form of F which admits of being integrated in succession, $\alpha + \beta \dots + \nu$ times.

Minster Yard, Lincoln, Aug. 30, 1843.

VII.—ON THE MOTION OF A PISTON AND OF THE AIR IN A CYLINDER.

By G. G. Stokes, B.A. Fellow of Pembroke College.

When a piston is in motion in a cylinder which also contains air, if the motion of the piston be not very rapid, so that its velocity is inconsiderable compared with the velocity of propagation of sound, the motions of the air may be divided into two classes, the one consisting of those which depend directly on the motion of the piston, the other, of those which are propagated with the velocity of sound, and depend on the initial state of the air, or on a breach of continuity in the

motion of the piston. If we suppose the initial velocity and condensation of the air in each section of the cylinder to be given, and also the initial velocity of the piston, both kinds of motion will in general take place, and the solution of the problem will be complicated. If, however, we restrict ourselves to motions of the first class, the approximate solution, though rather long, will be simple. In this case we must suppose the initial velocity and condensation of the air not to be given arbitrarily, but to be connected, according to a certain law which is yet to be found, with the motion of the piston. The problem as so simplified may perhaps be of some interest, as affording an example of the application of the partial differential equations of fluid motion, without requiring the employment of that kind of analysis which is necessary in most questions of that sort. It is, moreover, that motion of the air which it is proposed to consider, which principally affects the motion of the piston.

Conceive an air-tight piston to move in a cylinder which is closed at one end, and contains a mass of air between the closed end and the piston. For more simplicity, suppose the rest of the cylinder to contain no air. Let a point in the closed end be taken for origin, and let x be measured along the cylinder. Let x_1 be the abscissa of the piston; a the initial value of x_1; u the velocity parallel to x of any particle of air whose abscissa is x; p the pressure, ρ the density about that particle; Π the initial mean pressure; p_1 the value of p when $x = x_1$; X a function of x, the accelerating force acting on the air; then for the motion of the air we have

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dx} &= X - \frac{du}{dt} - u\frac{du}{dx}, \\ \frac{d\rho}{dt} &+ \frac{d\rho u}{dx} = 0, \\ \text{and} \qquad p &= k\rho, \end{aligned}\right\} \quad \ldots\ldots\ldots \ (1),$$

neglecting the variation of temperature.

We have also the conditions

$$u = 0 \quad \text{when} \quad x = 0 \ \ldots\ldots\ldots\ldots \ (2);$$

$$u = \frac{dx_1}{dt} \quad \text{when} \quad x = x_1 \ \ldots\ldots\ldots\ldots \ (3),$$

for positive values of t, and

$$\int_0^a p dx = \Pi a \quad \text{when} \quad t = 0 \ \ldots\ldots\ldots \ (4).$$

Eliminating ρ from equations (1), we have

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{k}\left(X - \frac{du}{dt} - u\frac{du}{dx}\right) \dots\dots\dots (5);$$

$$\frac{dp}{dt} + \frac{dpu}{dx} = 0 \dots\dots\dots\dots\dots (6).$$

Now, k being very large, for a first approximation let $\frac{1}{k}$ be neglected; then, integrating (5),

$$p = \phi(t).$$

Substituting in (6), and integrating,

$$u = \psi(t) - \frac{\phi'(t)}{\phi(t)}x.$$

The conditions (2) and (3) give

$$\psi(t) = 0; \quad \frac{\phi'(t)}{\phi(t)} = -\frac{1}{x_1}\frac{dx_1}{dt};$$

whence $$\phi(t) = \frac{C}{x_1}.$$

Substituting in (4) the value of p when $t = 0$, we have

$$\int_0^a \frac{C}{a}dx = C = \Pi a;$$

whence $$p = \Pi\frac{a}{x_1}; \quad u = \frac{x}{x_1}\frac{dx_1}{dt}.$$

Let now, for a second approximation,

$$p = \Pi\frac{a}{x_1} + p'; \quad u = \frac{x}{x_1}\frac{dx_1}{dt} + u';$$

so that p' and u' are small quantities of the order $\frac{1}{k}$; then, substituting these values in (5) and (6), remembering that the quantities which are not small must destroy each other, and retaining only small quantities of the first order, we have

$$\frac{dp'}{dx} = \frac{\Pi a}{kx_1}\left(X - \frac{x}{x_1}\frac{d^2x_1}{dt^2}\right) \dots\dots\dots (7);$$

$$\frac{dp'}{dt} + \frac{1}{x_1}\frac{dx_1}{dt}\frac{dp'x}{dx} + \Pi\frac{a}{x_1}\frac{du'}{dx} = 0 \dots\dots (8);$$

and the conditions (2), (3) and (4) give

$u' = 0$ when $x = 0$, or $x = x_1$, and t is positive. . . .(9);

$$\int_0^a p'dx = 0 \text{ when } t = 0 \dots\dots\dots (10).$$

Integrating (7), we have

$$p' = \frac{\Pi a}{kx_1}\left(\int_0^x X dx - \frac{x^2}{2x_1}\frac{d^2x_1}{dt^2}\right) + \omega(t) \ldots\ldots (11).$$

Substituting the values of p' and of its differential coefficients in (8), and integrating, we obtain

$$u' = \frac{x^3}{6kx_1^2}\frac{d}{dt}\left(x_1\frac{d^2x_1}{dt^2}\right) - \frac{1}{kx_1}\frac{dx_1}{dt}\int_0^x Xxdx - \frac{x}{\Pi a}\frac{d}{dt}\{x_1\omega(t)\} + \zeta(t)$$
$$\ldots\ldots(12).$$

The conditions (9) give $\zeta(t) = 0$;

$$\frac{1}{6k}\frac{d}{dt}\left(x_1\frac{d^2x_1}{dt^2}\right) - \frac{1}{kx_1^2}\frac{dx_1}{dt}\int_0^{x'} Xxdx - \frac{1}{\Pi a}\frac{d}{dt}\{x_1\omega(t)\} = 0;$$

and integrating, we get

$$x_1\omega(t) = \frac{\Pi a x_1}{6k}\frac{d^2x_1}{dt^2} - \frac{\Pi a}{k}\int_a^{x_1}\left(\int_0^{x_1} Xxdx\right)\frac{dx_1}{x_1^2} + C\ldots(13).$$

Putting f for the initial value of $\frac{d^2x_1}{dt^2}$ we have, from (10) and (11),

$$\frac{\Pi}{k}\left(\int_0^a dx\int_0^x X dx - \frac{fa^2}{b}\right) + \omega(0)\,a = 0;$$

and substituting the value of $\omega(0)$ given by this equation in (13), after having made $t = 0$, $x_1 = a$, $\frac{d^2x_1}{dt^2} = f$ in the latter, we have

$$C = -\frac{\Pi}{k}\int_0^a dx\int_0^x X dx.$$

Substituting this value of C in that of $\omega(t)$, and substituting in (11) and (12), and then substituting the values of p' and u' in those of p and u, we have

$$p = \Pi\frac{a}{x_1} + \frac{\Pi a}{kx_1}\left(\int_0^x X dx - \frac{x^2}{2x_1}\frac{d^2x_1}{dt^2}\right)$$
$$+ \frac{\Pi a}{6k}\frac{d^2x_1}{dt^2} - \frac{\Pi a}{kx_1}\int_a^{x_1}\left(\int_0^{x_1} Xxdx\right)\frac{dx_1}{x_1^2} - \frac{\Pi}{kx_1}\int_0^a\left(\int_0^x X dx\right)dx$$
$$\ldots.(14);$$

$$u = \frac{x}{x_1}\frac{dx_1}{dt} - \frac{x}{6k}\left(1 - \frac{x^2}{x_1^2}\right)\frac{d}{dt}\left(x_1\frac{d^2x_1}{dt^2}\right)$$
$$+ \frac{1}{kx_1}\frac{dx_1}{dt}\left\{\frac{x}{x_1}\int_0^{x_1} Xxdx - \int_0^x Xxdx\right\}\ldots(15).$$

Let A be the area of a section of the cylinder, and let $\frac{\Pi A a}{k} = \mu$, so that μ is the mass of the air; then we have

$$p_1 A = \Pi A \frac{a}{x_1} - \frac{\mu}{3}\frac{d^2x_1}{dt^2}$$
$$+ \frac{\mu}{x_1}\int_0^{x_1} X dx - \frac{\mu}{x_1}\int_0^{x_1}\left(\int_0^{x_1} X x dx\right)\frac{dx_1}{x_1^2} - \frac{\mu}{a x_1}\int_0^a dx \int_0^x X dx.$$

If there were no motion, the term $-\frac{\mu}{3}\frac{d^2x_1}{dt^2}$ would disappear. But in that case the value of $p_1 A$, the pressure on the piston, might be deduced immediately from the equation of equilibrium of an elastic fluid

$$\frac{1}{p}\frac{dp}{dx} = \frac{X}{k}.$$

Integrating this equation, determining the constant by the condition that $\int_0^{x_1} p dx = \Pi a$, multiplying by A, and putting $x = x_1$, we have, neglecting $\frac{1}{k^2}$,

$$p_1 A = \Pi A \frac{a}{x_1} + \frac{\mu}{x_1}\int_0^{x_1} X dx - \frac{\mu}{x_1^2}\int_0^{x_1}\left(\int_0^x X dx\right)dx.$$

Comparing this expression with the above, when the second term of the latter is left out, we have

$$\int_a^{x_1}\left(\int_0^{x_1} X x dx\right)\frac{dx_1}{x_1^2} + \frac{1}{a}\int_0^a dx \int_0^x X dx = \frac{1}{x_1}\int_0^{a_1} dx \int_0^a X dx,$$

a formula which may also be proved directly. We have then

$$p_1 A = \Pi A \frac{x}{x_1} - \frac{\mu}{3}\frac{d^2x_1}{dt^2} + \mu \frac{d}{dx_1}\left(\frac{1}{x_1}\int_0^{x_1} dx_1 \int_0^{x_1} X dx\right).$$

The first term would be the value of the pressure on the piston if the air had no inertia and were acted on by no external forces; the second term is that due to the *inertia* of the air; the last term is that due to the external forces, and in the case of gravity expresses the effect of the *weight* of the air. If M be the mass of the piston, P the accelerating force parallel to x acting on it, not including the pressure of the air, its equation of motion is

$$\left(M + \frac{\mu}{3}\right)\frac{d^2x_1}{dt^2} = MP + \Pi A \frac{a}{x_1} + \mu \frac{d}{dx_1}\left(\frac{1}{x_1}\int_0^{x_1} dx_1 \int_0^{x_1} X dx\right) \ldots (16).$$

Hence the effect of the inertia of the air is to increase the

mass of the piston by one third of that of the air, without increasing the moving force acting on it. If we could integrate equation (16) twice, we should determine the arbitrary constants by means of the initial values of x_1 and $\frac{dx_1}{dt}$, and thus get x_1 in terms of t: then, substituting in (14) and (15), we should obtain p and u as functions of x and t.

If the cylinder be vertical and smooth and turned upwards, we have $P = X = -g$; and if, moreover, the motion be very small, putting $x_1 = a + y$, and neglecting y^2, we have

$$\left(M + \frac{\mu}{3}\right)\frac{d^2y}{dt^2} + \frac{\Pi A}{a}y = \Pi A - \left(M + \frac{\mu}{2}\right)g.$$

The term at the second side of this equation is by hypothesis small, and, if we suppose the mean value of x to be taken for a, it is zero. On this supposition $\Pi A = \left(M + \frac{\mu}{2}\right)g$, and the time of a small oscillation will be $2\Pi\sqrt{\frac{M+\frac{\mu}{3}}{M+\frac{\mu}{2}}\cdot\frac{a}{g}}$, which becomes, since μ^2 is neglected throughout, $2\Pi\left(1-\frac{\mu}{12M}\right)\sqrt{\frac{a}{g}}$.

The reader who wishes to see the complete solution of the problem, in the case where no forces act on the air, and the air and piston are at first at rest, may consult a paper of Lagrange's with additions made by Poisson in the *Journal de l'Ecole Polytechnique.*

VIII.—ON THE EQUATIONS OF THE MOTION OF HEAT REFERRED TO CURVILINEAR CO-ORDINATES.

LET x, y, z be the rectangular co-ordinates of any point in space, and let $\lambda, \lambda_1, \lambda_2$ be any functions of x, y, z, such that

$$\lambda = a, \quad \lambda_1 = a_1, \quad \lambda_2 = a_2 \ldots\ldots\ldots \quad (1)$$

are the equations of three surfaces cutting one another at right angles for any values of the variable parameters a, a_1, a_2. The three series produced by giving all possible values to these parameters form what is called a system of conjugate orthogonal surfaces.

It has been proved by Dupin that the surfaces of any two of the three series cut each surface of the other series in its

lines of curvature. Hence if one series be given, the other two are determinable from it, except in such extreme cases as those in which the lines of curvature of the given series are indeterminate.

In the method of curvilinear co-ordinates, as proposed by Lamé, the position of any point in space is determined by the three conjugate surfaces of the system (1) which intersect in the point. Thus, if any point P be given, there will be at least one of the surfaces of the first series which passes through it, and in general, but especially such cases as we shall consider, there will be only one. a_1 the value of λ corresponding to this surface is one of the co-ordinates of P. The other two co-ordinates are the values of λ_1 and λ_2 corresponding to the surfaces of the second and third series which contain the two lines of curvature through P of the first surface.

This general method of co-ordinates comprehends the two systems, rectangular and polar, in ordinary use. Thus, if $\lambda = x$, $\lambda_1 = y$, $\lambda_2 = z$, equations (1) will become

$$x = a, \quad y = a_1, \quad z = a_2,$$

the equations to three planes at right angles, by their intersection, determining the point P, whose rectangular co-ordinates are a, a_1, a_2. Similarly, if the first be a series of concentric spheres, the second a series of planes through a diameter of the sphere, and the third a series of cones having this diameter for axis, and the centre for vertex, $\lambda, \lambda_1, \lambda_2$ will be polar co-ordinates.

The equations of the motion of heat in a solid body may be referred to the general system of curvilinear co-ordinates in the following manner, which is exactly similar to that by which Fourier establishes them for rectangular rectilinear co-ordinates. For simplicity we shall suppose the body homogeneous, though the investigation would be in principle as simple if this were not the case. Let d be its density, h its conducting power, and c its capacity for heat, or the quantity of heat necessary to raise the temperature of a unit of its mass by unity.

Let $\lambda, \lambda_1, \lambda_2$ be the co-ordinates of any point P in the body, and let $\lambda + d\lambda$, $\lambda_1 + d\lambda_1$, $\lambda_2 + d\lambda_2$ be those of an adjacent point P'. The portions of the six surfaces corresponding to λ, $\lambda + d\lambda$, λ_1, $\lambda_1 + d\lambda_1$, λ_2, and $\lambda_2 + d\lambda_2$ adjacent to the points P and P' will form a rectangular parallelopiped, of which P and P' are opposite angles. Let dp, dp_1, dp_2 be the three edges of this parallelopiped which are respectively

perpendicular to the surfaces corresponding to λ, λ_1, and λ_2: dp will be the element of the normal to the surface λ, commencing at this surface and terminated by the surface $\lambda + d\lambda$, and similarly with dp_1 and dp_2. Hence, by a known theorem,

$$dp = \left(\frac{d\lambda^2}{dx^2} + \frac{d\lambda^2}{dy^2} + \frac{d\lambda^2}{dz^2}\right)^{-\frac{1}{2}} d\lambda = H d\lambda, \text{ for brevity.} \ldots (a),$$

$$dp_1 = \left(\frac{d\lambda_1^{\ 2}}{dx^2} + \frac{d\lambda_1^{\ 2}}{dy^2} + \frac{d\lambda_1^{\ 2}}{dz^2}\right)^{-\frac{1}{2}} d\lambda_1 = H_1 d\lambda_1 \ldots\ldots\ldots (b),$$

$$dp_2 = \left(\frac{d\lambda_2^{\ 2}}{dx^2} + \frac{d\lambda_2^{\ 2}}{dy^2} + \frac{d\lambda_2^{\ 2}}{dz^2}\right)^{-\frac{1}{2}} d\lambda_2 = H_2 d\lambda_2 \ldots\ldots\ldots (c).$$

Let v be the temperature of the body at P, and let $v + d_\lambda v + d_{\lambda_1} v + d_{\lambda_2} v$ be the temperature at P'; $d_\lambda v$, $d_{\lambda_1} v$, and $d_{\lambda_2} v$ denoting the increments of v which correspond to the increments $d\lambda$, $d\lambda_1$, $d\lambda_2$ of λ, λ_1, λ_2. This notation, for *partial differentials*, we shall find it convenient to use in those cases in which it is not convenient to employ *partial differential coefficients*. The quantity of heat which flows into the rectangular parallelopiped $dp \,.\, dp_1 \,.\, dp_2$ across the side which coincides with λ, in the time dt, is

$$-k \frac{d_\lambda v}{dp} dp_1 dp_2 \,.\, dt,$$

and the quantity which flows out across the opposite side, in the same time,

$$-k\left(\frac{d_\lambda v}{dp} dp_1 dp_2\right) dt - k d_\lambda \left(\frac{d_\lambda v}{dp} dp_1 dp_2\right) dt.$$

The difference of these two expressions,

$$\text{or } k d_\lambda \left(\frac{d_\lambda v}{dp} dp_1 dp_2\right) dt,$$

is the whole quantity of heat which flows into the element during the time dt in a direction perpendicular to λ.

Similarly, the quantities of heat which the element gains in the same time, by motion in directions perpendicular to λ_1 and λ_2, are

$$k d_{\lambda_1} \left(\frac{d_{\lambda_1} v}{dp_1} dp dp_2\right) dt, \quad \text{and} \quad k d_{\lambda_2} \left(\frac{d_{\lambda_2} v}{dp_2} dp dp_1\right) dt,$$

and therefore the entire quantity of heat gained by the element in the time dt, is

$$k\left\{d_\lambda \left(\frac{d_\lambda v}{dp} dp_1 dp_2\right) + d_{\lambda_1} \left(\frac{d_{\lambda_1} v}{dp_1} dp dp_2\right) + d_{\lambda_2} \left(\frac{d_{\lambda_2} v}{dp_2} dp dp_1\right)\right\};$$

or, by (*a*), (*b*), (*c*),

$$k\left\{\frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\frac{H_1H_2}{H}\right)+\frac{d}{d\lambda_1}\left(\frac{dv}{d\lambda_1}\frac{HH_2}{H_1}\right)+\frac{d}{d\lambda_2}\left(\frac{dv}{d\lambda_2}\frac{HH_1}{H_2}\right)\right\}d\lambda d\lambda_1 d\lambda_2,$$

where, in accordance with the ordinary notation of partial differential coefficients, the suffixes are omitted in the partial differentials.

Now if dv be the alteration of the temperature of the element $dpdp_1dp_2$, in the time dt, the corresponding addition of heat is

$$c\,.\,d\,.\,dv\,dp\,dp_1dp_2,\quad\text{or}\quad cd\,.\,dv\,.\,HH_1H_2d\lambda\,d\lambda_1d\lambda_2.$$

Hence we have the equation

$$HH_1H_2\frac{dv}{dt}=\frac{k}{c\,.\,d}\left\{\frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\frac{H_1H_2}{H}\right)+\frac{d}{d\lambda_1}\left(\frac{dv}{d\lambda_1}\frac{HH_2}{H_1}\right)+\frac{d}{d\lambda_2}\left(\frac{dv}{d\lambda_2}\frac{HH_1}{H_2}\right)\right\}\ldots\ldots.(2),$$

which expresses fully, by means of curvilinear co-ordinates, the motion of heat in the interior of homogeneous solid bodies.

If the motion has continued so long as to have become uniform, then $\frac{dv}{dt}=0$, and the equation becomes

$$\frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\frac{H_1H_2}{H}\right)+\frac{d}{d\lambda_1}\left(\frac{dv}{d\lambda_1}\frac{HH_2}{H_1}\right)+\frac{d}{d\lambda_2}\left(\frac{dv}{d\lambda_2}\frac{HH_1}{H_2}\right)=0\ldots(3).$$

This equation was first given by Lamé, in a paper entitled "*Mémoire sur les lois de l'Equilibre du Fluide Ethéré*," in the *Journal de l'Ecole Polytechnique*, (Vol. III. cahier XXIII.), who deduced it by a very laborious transformation from the equation

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=0\ \ldots\ldots\ldots\ldots\ (4),$$

in which the motion is referred to rectilinear co-ordinates. Equation (3) comprehends, as particular forms, equation (4), and the equation in which the motion is referred to polar co-ordinates. The former of these is obtained at once, if we put $\lambda=x$, $\lambda_1=y$, $\lambda_2=z$, which gives $H=H_1=H_2=1$. To find the latter, let

$$\lambda=(x^2+y^2+z^2)^{\frac{1}{2}}=r,$$

$$\lambda_1=\tan^{-1}\frac{y}{x}=\phi,$$

$$\lambda_2=\frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}}=\cos\theta.$$

Hence

$$H = \left(\frac{dr^2}{dx^2} + \frac{dr^2}{dy^2} + \frac{dr^2}{dz^2}\right)^{-\frac{1}{2}} = 1,$$

$$H_1 = \left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)^{-\frac{1}{2}} = r \sin \theta,$$

$$H_2 = \left\{\frac{(d\cos\theta)^2}{dx^2} + \frac{(d\cos\theta)^2}{dy^2} + \frac{(d\cos\theta^2)}{dz^2}\right\}^{-\frac{1}{2}} = \frac{r^2}{(r^2 - z^2)^{\frac{1}{2}}} = \frac{r}{\sin\theta}.$$

Therefore (3) becomes

$$\frac{d}{dr}\left(r^2 \frac{dv}{dr}\right) + \frac{d}{d\cos\theta}\left(\sin^2\theta \frac{dv}{d\cos\theta}\right) + \frac{d}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{dv}{d\phi}\right) = 0,$$

$$\text{or} \quad r\frac{d^2(rv)}{dr^2} + \frac{d}{d\cos\theta}\left(\sin^2\theta\frac{dv}{d\cos\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2v}{d\phi^2} = 0,$$

the well-known equation of which so much use is made in the determination of the properties of the expansion of v in a series of powers of r. Equation (3) may also be applied to find the conditions which must be satisfied so that each series of a system of conjugate orthogonal surfaces may be isothermal; and we shall thus be enabled to answer the question proposed in a paper in the last number of this *Journal* (vol. III. p. 286).

If the first series of surfaces, or the series represented by the equation $\lambda = a$, be isothermal, (3) must be satisfied by the assumption $v = f(\lambda)$, and therefore

$$\frac{d}{d\lambda}\left(\frac{H_1 H_2}{H} f'\lambda\right) = 0 \;\ldots\ldots\ldots\ldots (5).$$

Similarly, if the second and third series be isothermal,

$$\frac{d}{d\lambda_1}\left(\frac{HH_2}{H_1} f'_1\lambda_1\right) = 0 \;\ldots\ldots\ldots\ldots (6),$$

$$\frac{d}{d\lambda_2}\left(\frac{HH_1}{H_2} f'_2\lambda_2\right) = 0 \;\ldots\ldots\ldots\ldots (7).$$

Integrating these equations, we have

$$\frac{H_1H_2}{H} f'\lambda = F(\lambda_1, \lambda_2) \ldots\ldots\ldots\ldots (8),$$

$$\frac{HH_2}{H_1} f'_1\lambda_1 = F_1(\lambda, \lambda_2) \ldots\ldots\ldots\ldots (9),$$

$$\frac{HH_1}{H_2} f'_2\lambda_2 = F_2(\lambda, \lambda_1) \ldots\ldots\ldots (10).$$

These three equations, together with the following,

$$\frac{d\lambda_1}{dx}\frac{d\lambda_2}{dx}+\frac{d\lambda_1}{dy}\frac{d\lambda_2}{dy}+\frac{d\lambda_1}{dz}\frac{d\lambda_2}{dz}=0 \;\ldots\ldots\; (11),$$

$$\frac{d\lambda_2}{dx}\frac{d\lambda}{dx}+\frac{d\lambda_2}{dy}\frac{d\lambda}{dy}+\frac{d\lambda_2}{dz}\frac{d\lambda}{dz}=0 \;\ldots\ldots\; (12),$$

$$\frac{d\lambda}{dx}\frac{d\lambda_1}{dx}+\frac{d\lambda}{dy}\frac{d\lambda_1}{dy}+\frac{d\lambda}{dz}\frac{d\lambda_1}{dz}=0 \;\ldots\ldots\; (13),$$

which make the surfaces cut one another at right angles, are the conditions which must be satisfied if the three series be orthogonal and isothermal. If we could eliminate all the quantities depending on two of the series, the second and third for instance, we should find two equations relative to the first series which make it both be isothermal itself, and possess the property that the two series which cut its surfaces orthogonally shall also be isothermal. This elimination is probably quite impracticable in general. There is however an extensive class of surfaces which we see by inspection satisfies each condition, the class of cylindrical isothermal surfaces. For, if the axis of z be parallel to the generating lines of a series of isothermal cylindrical surfaces represented by the equation $\lambda = a$, λ will be independent of z, and $\lambda_1 = a_1$ being the equation of a series of orthogonal cylindrical surfaces, λ_1 will also be independent of z. The third series of the conjugate system will be a series of planes parallel to xy. Hence H and H_1 are functions of x and y alone, and therefore of λ and λ_1, and H_2 is a function of z and therefore of λ_2. Now since $\lambda = a$ is an isothermal system, (5) must be satisfied, and therefore $\dfrac{H_1H_2}{H}$ must contain the factor $\dfrac{1}{f'(\lambda)}$, and λ must enter in no other manner. Hence $\dfrac{H_1}{H}$ is the product of two functions, one of λ and the other of λ_1, and therefore $\dfrac{HH_2}{H_1}$ must consist of three factors, functions of λ, λ_1, λ_2 separately. Hence if we choose the arbitrary function $f'(\lambda_1)$ properly, (6) will be satisfied. Also, in every case, whether $\lambda = a$ be isothermal or not, we may choose $f'_2(\lambda_2)$ in such a manner that (7) shall be satisfied; in fact a series of parallel planes is necessarily isothermal. We have thus seen that the two conjugate orthogonal series to a series of isothermal cylinders are themselves isothermal, which agrees with what was proved in the paper already alluded to, (vol. III. p. 286).

In the case in which λ and λ_1 are surfaces of revolution we can obtain from (5), (6), and (7) a simple condition, which being satisfied, λ and λ_1 will each be isothermal, if one of them is so. To effect this, let the axis of revolution be taken for axis of x, and let $\rho = (y^2+z^2)^{\frac{1}{2}}$. The generating lines of the surfaces of revolution of which the series λ and λ_1 are composed will be plane curves expressed by two equations between x and ρ. If in these equations we write $(y^2+z^2)^{\frac{1}{2}}$ for ρ, the results will be the equations $\lambda = a$, $\lambda_1 = a_1$ of the surfaces of revolution. Hence λ and λ_1 are functions of x and ρ, and therefore, conversely, x and ρ are functions of λ and λ_1. Hence we see readily that H and H_1 depend only on x and ρ, or on λ and λ_1. Also, the equation $\lambda_2 = a_2$ must represent a series of planes passing through the axis of x, and we may therefore assume $\lambda_2 = \omega = \tan^{-1}\dfrac{y}{z}$. From this we readily deduce $H_2 = \rho$. Hence (7) is always satisfied, independently of λ and λ_1; that is to say, a series of planes passing through a fixed axis, is isothermal. Hence we have only to consider the conditions relative to λ and λ_1, or equations (5) and (6). If we substitute for H_2 its value, these become

$$\frac{d}{d\lambda}\left\{\frac{\rho H_1}{H} f'(\lambda)\right\} = 0 \qquad (14),$$

$$\frac{d}{d\lambda_1}\left\{\frac{\rho H}{H_1} f'(\lambda_1)\right\} = 0 \qquad (15).$$

From these we readily deduce

$$\rho = F(\lambda).F_1(\lambda_1) \qquad (16),$$

F and F_1 being entirely arbitrary. Hence, if the equation $\lambda = a$ of a series of surfaces of revolution be given and $\lambda_1 = a_1$, the equation to the orthogonal system be deduced, and if it be found that the distance of any point from the axis of revolution can be expressed in the form (13), then both systems, or neither, will be isothermal.

If the given series be isothermal this test may be applied very readily, as in that case we are always able at once to find the equation of the conjugate series. To shew this, let the series λ be isothermal. Then (14) will be true, and therefore we have, by integration,

$$\frac{\rho H_1}{H} f'(\lambda) = F(\lambda_1),$$

$$\therefore \quad H_1 = \frac{HF(\lambda_1)}{\rho f'(\lambda)} \ldots\ldots\ldots\ldots\ldots (17).$$

Also, since the sections of the two surfaces made by any plane through x are perpendicular to one another, we have

$$\frac{d\lambda}{dx}\frac{d\lambda_1}{dx} + \frac{d\lambda}{d\rho}\frac{d\lambda_1}{d\rho} = 0 \ldots\ldots\ldots\ldots (18);$$

this equation gives

$$\frac{\dfrac{d\lambda_1}{dx}}{\dfrac{d\lambda}{d\rho}} = -\frac{\dfrac{d\lambda_1}{d\rho}}{\dfrac{d\lambda}{dx}} = \frac{H}{H_1},$$

and therefore, by (14),

$$F(\lambda_1)\frac{d\lambda_1}{dx} = \frac{dL_1}{dx} = \rho f'(\lambda)\frac{d\lambda}{d\rho},$$

$$F(\lambda_1)\frac{d\lambda_1}{d\rho} = \frac{dL_1}{d\rho} = -\rho f'(\lambda)\frac{d\lambda}{dx},$$

if $L_1 = \int F(\lambda_1)\,d\lambda_1$; therefore

$$L_1 = \int \rho f'(\lambda)\left(\frac{d\lambda}{d\rho}dx - \frac{d\lambda}{dx}d\rho\right) \ldots\ldots\ldots (19),$$

and $L_1 = a$ is the equation of the series of orthogonal trajectories to the series of curves in which the surfaces of revolution of the given isothermal system are cut by any plane through the axis. We may verify this solution by observing that the criterion of integrability for the expressions given above for $\dfrac{dL_1}{dx}$ and $\dfrac{dL_1}{d\rho}$ is satisfied; since, if we transform the equation

$$\frac{d^2(f\lambda)}{dx^2} + \frac{d^2(f\lambda)}{dy^2} + \frac{d^2(f\lambda)}{dz^2} = 0,$$

to the independent variables x and ρ, $f\lambda$ being independent of $\dfrac{y}{z}$, we have

$$f'(\lambda)\left(\frac{d^2\lambda}{dx^2} + \frac{d^2\lambda}{d\rho^2}\right) + f''(\lambda)\left(\frac{d\lambda^2}{dx^2} + \frac{d\lambda^2}{d\rho^2}\right) + \frac{1}{\rho}f'(\lambda)\frac{d\lambda}{d\rho} = 0.$$

If λ itself satisfy (4), we may take $f(\lambda) = \lambda$, and the transformed equation becomes

$$\frac{d^2\lambda}{dx^2} + \frac{d^2\lambda}{d\rho^2} + \frac{1}{\rho}\frac{d\lambda}{d\rho} = 0 \ldots\ldots\ldots\ldots (20).$$

Also, if we take $\lambda_1 = L_1$, (19) becomes

$$\lambda_1 = \int \rho \left(\frac{d\lambda}{d\rho} dx - \frac{d\lambda}{dx} d\rho \right) \dots\dots\dots (21).$$

As an example of the application of these formulæ, let λ represent the series of *surfaces of equilibrium* in the case of a sphere having matter distributed over it, according to the law of the distribution of electricity on a neutral conducting sphere under the influence of a distant electrified body. It is readily shown that, if λ be proportional to the *potential* of such a system, on external points, and if the centre of the sphere be origin, and the line joining this point and the influencing body axis of x, we have

$$\lambda = \frac{x}{(x^2 + \rho^2)^{\frac{3}{2}}} = \frac{x}{r^3} \dots\dots\dots\dots (a).$$

Hence $\lambda = a$ represents a series of isothermal surfaces of revolution, and λ satisfies (20), as may be readily verified. Hence, if $\lambda_1 = a_1$ be the orthogonal system of surfaces of revolution, we have, by (21),

$$\lambda_1 = \frac{\rho^2}{(x^2 + \rho^2)^{\frac{3}{2}}} = \frac{\rho^2}{r^3} \dots\dots\dots\dots\dots (b).$$

If between (a) and (b) we eliminate x, we have

$$\rho^{\frac{4}{3}} + \frac{\lambda_1^{\ 2}}{\lambda^2} \rho^{\frac{2}{3}} - \frac{\lambda_1^{\frac{4}{3}}}{\lambda^2} = 0 \dots\dots\dots\dots (c).$$

The value of ρ deduced from this equation is not of the form $F(\lambda) . F_1(\lambda_1)$; and hence, by (16), the second series is not isothermal. Similarly, if λ represent the potential of two equal material points situated at the distance $2a$ from one another on the point xyz, we have

$$\lambda = \frac{1}{\{(x-a)^2 + \rho^2\}^{\frac{1}{2}}} + \frac{1}{\{(x+a)^2 + \rho^2\}^{\frac{1}{2}}} \dots\dots (d).$$

Then, by (18), we have, for the orthogonal system,

$$\lambda_1 = \frac{x-a}{\{(x-a)^2 + \rho^2\}^{\frac{1}{2}}} + \frac{x+a}{\{(x+a)^2 + \rho^2\}^{\frac{1}{2}}} \dots\dots (e).$$

The value of ρ deduced from these two equations cannot, I think, be of the form $F(\lambda)$, $F(\lambda_1)$, and hence in this case also the second series is not isothermal. Hence the question proposed in the paper already referred to (vol. III. p. 286)

must be answered in the negative, the proposition to which it refers being not generally true, since we have found cases of surfaces of revolution with regard to which it does not hold; but it holds with regard to every system of conjugate orthogonal surfaces, for which equations (8), (9), and (10) are satisfied. Also, since it does not appear that any two of these equations imply the third, it may happen that two series of a system may be isothermal and the third not, as is exemplified in the particular cases above considered, in each of which a series of surfaces of revolution and of orthogonal planes are isothermal, and the third series, consisting of surfaces of revolution, is not isothermal.

P. Q. R.

IX.—OF ASYMPTOTES TO ALGEBRAIC CURVES.

By D. F. Gregory, M.A. Fellow of Trinity College.

The ordinary method of deducing the equations to asymptotes to plane curves, whether by finding finite values of the intercepts of the tangents for infinite values of the co-ordinates of the point of contact, or by the more convenient method of expansion in descending powers of one of the variables, are essentially unsymmetrical. Moreover the former is often inapplicable from the difficulty of finding the true value of a fraction of which the numerator and denominator are infinite, and the latter fails when the asymptote is parallel to one of the co-ordinate axes. The following method, though appropriate to algebraic curves only, is for them of very easy application, and, besides being symmetrical, leads us readily to the demonstrations of various properties of asymptotes to curves.

Let the equation to the curve, cleared of fractions and radicals, be put in the form

$$u = \phi_n(x, y) + \phi_{n-1}(x, y) + \phi_{n-2}(x, y) + \&c. + \phi_0 = 0 \ldots (1),$$

the symbol $\phi_r(x, y)$ denoting generally a *homogeneous* function of r dimensions in x and y; then the equation to the tangent at a point (x, y) may be written

$$x'\frac{du}{dx} + y'\frac{du}{dy} + \phi_{n-1}(x, y) + 2\phi_{n-2}(x, y) + \&c. + n\phi_0 = 0 \ldots (2),$$

x', y' being the current co-ordinates of the tangent.

The definition of an asymptote is, that it is a line which, passing through a point at a finite distance from the origin, touches the curve at an infinite distance. Now if x, y be the co-ordinates of the point of contact; x', y' those of a point

through which the line passes; l, m the cosines of the angles which the line makes with the axes, we have

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r\ldots\ldots\ldots\ldots\text{(3)},$$

r being the distance between the point x, y and x', y'. Hence

$$x=x'+lr, \quad y=y'+mr,$$

and substituting these values in (1), it becomes

$$\phi_n(l,m)r^n+\left\{\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l,m)+\phi_{n-1}(l,m)\right\}r^{n-1}+\&\text{c.}=0\ldots\text{(4)}.$$

But if the point x, y is at an infinite distance, r_1 must be infinite, which involves the condition that the coefficient of the highest power of r shall vanish; consequently we must have

$$\phi_n(l, m)=0\ldots\ldots\ldots\ldots\ldots\text{(5)}$$

as an equation for determining the *direction* of the asymptotes. Again, if we substitute in (2) for x and y their values derived from (3), we have, arranging in powers of r,

$$\left\{\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l, m)+\phi_{n-1}(l, m)\right\}r^{n-1}+\&\text{c.}=0.\ldots\text{(6)}.$$

Since the asymptote is by definition a particular case of the tangent, this equation also must give an infinite value for r, which involves the condition

$$\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l, m)+\phi_{n-1}(l, m)=0\ldots\ldots\text{(7)}.$$

Now this equation is linear in x' and y', which are the co-ordinates of *any* point through which the asymptote passes, that is of any point in the line; so that this equation is in fact the equation to the asymptote, if we substitute in it the relations between l and m, which satisfy equation (5). As from the homogeneity of the terms, l and m finally disappear, and as they are involved exactly as x and y are in the equation to the curve, we may express the process for finding the asymptotes to a curve simply as follows. Let the equation to the curve be put in the form

$$u_n+u_{n-1}+u_{n-2}+\&\text{c.}+u_0=0\ldots\ldots\ldots\text{(8)},$$

u_r being a homogeneous function in x and y of the r^{th} order, then the equation to the asymptotes will be found by eliminating x and y from

$$x'\frac{du_n}{dx}+y'\frac{du_n}{dy}+u_{n-1}=0$$

by means of the relation between x and y given by the equation

$$u_n = 0.$$

Since the equation $u_n = 0$ is of the n^{th} degree in x and y, it appears that a curve of the n^{th} degree can have n asymptotes and no more. If there be any impossible factors in $u_n = 0$, there are no corresponding asymptotes; and as impossible factors enter the equation by pairs, a curve of the n^{th} degree must have n or $n - 2$ or $n - 4$ or &c. asymptotes.

As an example take the curve

$$y^3 - x^3 - ax^2 = 0.$$

In this case $u_n = 0$ becomes $y^3 - x^3 = 0$, in which there is only one possible factor $y - x = 0$. The other equation

$$3y'y^2 - 3x'x^2 - ax^2 = 0$$

becomes, on putting $y = x$,

$$y' - x' = \frac{a}{3},$$

which is the equation to the asymptote.

Again, let the curve be

$$xy^2 - x^3 + 2a^2y = 0;$$

then the equation $u_n = 0$ becomes

$$xy^2 - x^3 = 0,$$

giving three factors

$$x = 0, \quad y = +x, \quad y = -x.$$

The first of these substituted in the equation

$$x'(y^2 - 3x^2) + 2y'xy = 0,$$

gives $\qquad x' = 0,$

the second and third give

$$x' - y' = 0 \quad \text{and} \quad x' + y' = 0,$$

which are the equations to the three asymptotes, the first being evidently the axis of y, and the others inclined at angles $\pm 45°$ to the axis of x. Let the curve be

$$(x + a)y^2 = (y + b)x^2, \quad \text{or} \quad xy^2 - yx^2 + ay^2 - bx^2 = 0.$$

The equation $u_n = 0$ here becomes

$$xy^2 - yx^2 = 0, \quad \text{or} \quad xy(y - x) = 0,$$

which has three possible factors

$$x = 0, \quad y = 0, \quad y - x = 0.$$

The general equation to the asymptotes is

$$x'(y^2 - 2xy) + y'(2xy - x^2) + ay^2 - bx^2 = 0:$$

for $x = 0$ this is reduced to

$$x' + a = 0\,;$$

for $y = 0$ it becomes $\qquad y' + b\,;$

and for $y - x = 0$ it becomes

$$y' - x' + a - b = 0,$$

and these are the equations to the three asymptotes. One advantage of this method of finding asymptotes is that a simple inspection of the highest terms of the equation shows at once the number and direction if the asymptotes. The method however fails if the equation $u_n = 0$ contain equal possible roots, indicating the existence of parallel asymptotes. For in this case $\frac{du_n}{dx}$ and $\frac{du_n}{dy}$ will vanish along with u_n, and consequently the equation to the asymptote is nugatory; but a slight extension of the process enables us to overcome the difficulty. It will be seen that the two equations are the coefficients of the n^{th} and $(n-1)^{\text{th}}$ power of r in the expansion (4). In the case of failure from the existence of m equal roots in the equation $u_n = 0$, all that is necessary is to equate to zero the coefficient of the highest power of r which is unaffected by the equality of roots that is the coefficient of the $(n-m)^{\text{th}}$ power of r, and this equation combined with the m equal roots of $u_n = 0$ gives the equations to the parallel asymptotes. If there be two equal roots, and the equation to the curve be put in the form (8), the equation for determining the parallel asymptotes, or the coefficient of r^{n-2} equated to zero, is

$$\frac{1}{2}\left(x'^2\frac{d^2u_n}{dx^2} + 2x'y'\frac{d^2u_n}{dxdy} + y'^2\frac{d^2u_n}{dy^2}\right) + x'\frac{du_{n-1}}{dx} + y'\frac{du_{n-1}}{dy} + u_{n-2} = 0 \ldots (9).$$

As an example take the curve

$$x^2y - x^3 - 3bxy + 2b^2y = 0.$$

Here $u_n = 0$ becomes $x^2y - x^3 = 0$ gives

$$x^2 = 0, \qquad y - x = 0.$$

$$\frac{d^2u_n}{dx^2} = 2y - 6x = 2y \text{ when } x = 0,$$

$$\frac{d^2u_n}{dxdy} = 2x = 0 \text{ when } x = 0, \qquad \frac{d^2u_n}{dy^2} = 0\,;$$

$$\frac{du_{n-1}}{dx} = -\,3by, \qquad \frac{du_{n-1}}{dy} = -\,3bx = 0 \text{ when } x = 0\,;$$

consequently equation (9) becomes, on dividing by y,

$$x'^2 - 3bx' + 2b^2 = 0,$$

which is decomposable into the two factors

$$x' - b = 0, \qquad x' - 2b = 0,$$

the equations to two asymptotes parallel to the axis of y. The equation to the other asymptote is given by the equation $y - x = 0$ is

$$y' - x' - 3b = 0.$$

In like manner we may shew that the curve

$$x^2 (x^2 + y^2) = a^2 (y - x)^2$$

has two parallel asymptotes, of which the equations are

$$x' + a = 0, \text{ and } x' - a = 0.$$

The equation (4) for any values of l and m is an equation for determining the value of r, the portion of the line whose direction-cosines are l and m intercepted between a given point and the curve. It is generally of n dimensions, so that the line generally meets the curve in n points; but when the line is an asymptote, the first two terms disappear and the equation is reduced to $n - 2$ dimensions. Consequently an asymptote cannot meet its curve in more than $n - 2$ points; and as for all lines parallel to an asymptote the first term of (4) vanishes, lines parallel to an asymptote cannot meet the curve in more than $n - 1$ points.

Since, from what precedes, it appears that the equation to an asymptote depends only on the terms involving the highest and second highest powers of the variables, all curves for which these are the same have the same asymptotes, and *vice versâ*. And as among the curves of the n^{th} order is to be included that made up of the n asymptotes themselves, the product of their n linear equations must have the same highest and second highest terms as the equation to the curve; that is, the equation to the curve differs from the product of the equations to the n asymptotes only in terms of the $(n - 2)^{\text{th}}$ order: so that if the equations to the asymptotes be the n linear equations

$$u' = 0, \quad u'' = 0 \; \ldots\ldots \; u^{(n)} = 0,$$

that to the curves to which these are asymptotes may be written

$$u'u'' \ldots . u^{(n)} + u_{n-2} + u_{n-3} + \ldots . + u_0 = 0.$$

Thus if the curve be of the third order, its equation is

$$u'u''u''' + u_1 + u_0 = 0.$$

When an asymptote meets the curve, which it can do in one point only, this is to be combined with $u' = 0$, $u'' = 0$, $u''' = 0$, any one of which reduces the preceding equation to

$$u_1 + u_0 = 0,$$

which being a linear equation common to the three points in which the curve meets its asymptotes, shews that they lie in one straight line.

In like manner we may shew that the six points in which the curve is cut by lines drawn parallel to the asymptotes all lie in a curve of the second order.

X.—MATHEMATICAL NOTES.

1. If a plane passes through any point of a surface, and makes any function of the intercepts it cuts off from the axes, a maximum or a minimum when it touches the surface, this maximum or minimum value is constant for all points of the surface; and, conversely, if for every point of a surface, a given function of the intercepts of the tangent plane is constant, this function is, with reference to any single point of the surface, a maximum or minimum for the tangent plane.

This appears at once from the following considerations: x, y, z being a point in the surface, x_0, y_0, z_0, the three intercepts, $\phi(x_0, y_0, z_0)$ the given function, if we seek to determine the surface so that ϕ shall be a maximum or minimum, we have the equations

$$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} = 1 \; \ldots\ldots\ldots\ldots \; (1),$$

$$\frac{d\phi}{dx_0} = \mu\,\frac{x}{x_0^2} \; \ldots. \; (2), \qquad \frac{d\phi}{dy_0} = \mu\,\frac{y}{y_0^2} \; \ldots. \; (3);$$

$$\frac{d\phi}{dz_0} = \mu\,\frac{z}{z_0^2} \; \ldots\ldots. \; (4),$$

μ being a factor. From these equations we get

$$x_0 = f_1(xyz), \qquad y_0 = f_2(xyz), \qquad z_0 = f_3(xyz),$$

and therefore the differential equation of the surface is

$$\frac{dx}{f_1(xyz)} + \frac{dy}{f_2(xyz)} + \frac{dz}{f_3(xyz)} = 0 \; \ldots\ldots. \; (5);$$

for by the ordinary equation of the tangent plane we have

$$\frac{1}{x_0} : \frac{1}{y_0} : \frac{1}{z_0} :: \frac{dF}{dx} : \frac{dF}{dy} : \frac{dF}{dz},$$

$F = 0$ being the equation to the surface.

Again, if we seek to determine the surface, so that ϕ shall be constant, *i.e.* to find the envelope of all the planes represented by (1), we have (1), (2), (3), (4), as before, and in addition

$$\phi(x_0, y_0, z_0) = c \; \ldots\ldots. \; (6).$$

Thus, as before, $x_0 = f_1$, $y_0 = f_2$, $z_0 = f_3$, and the equation of the surface may be got by integrating (5), and determining the constant so that the result may coincide with (6). And the identity of the equations connecting x_0, y_0, z_0, and x, y, z, in the two cases proves our proposition and its converse. Take as an example the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad \text{Here } x_0 = \frac{a^2}{x}, \quad y_0 = \frac{b^2}{y}, \quad z_0 = \frac{c^2}{z}.$$

$$\therefore \frac{a^2}{x_0^2} + \frac{b^2}{y_0^2} + \frac{c^2}{z_0^2} = 1.$$

Therefore the tangent plane to any point of the ellipsoid makes $\frac{a^2}{x_0^2} + \frac{b^2}{y_0^2} + \frac{c^2}{z_0^2}$, a minimum with reference to any plane passing through that point.

(ϵ).

2. To find the value of

$$\frac{fa_1}{(a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)} + \frac{fa_2}{(a_1 - a_2)(a_3 - a_2)\ldots(a_n - a_2)} + \&c. = A,$$

when $a_1 = a_2 = \&c. = a$.

Let $a_1 = a + z_1$, $a_2 = a + z_2$, &c.

$$\therefore (a_2 - a_1)(a_3 - a_1)\ldots.(a_n - a_1) = (z_2 - z_1)(z_3 - z_1)\ldots.(z_n - z_1),$$

and so of the rest;

$$\therefore A = fa \left\{ \frac{1}{(z_2 - z_1)\ldots.(z_n - z_1)} + \frac{1}{(z_1 - z_2)\ldots.(z_n - z_2)} + \&c. \right\}$$

$+ \&c.$

$$+ \frac{f^{(p)}a}{1.2\ldots p} \left\{ \frac{z_1^p}{(z_2 - z_1)\ldots.(z_n - z_1)} + \frac{z_2^p}{(z_1 - z_2)\ldots.(z_n - z_2)} + \&c. \right\}$$

$+ \&c.$ by Taylor's theorem.

Now, when $z_1 = z_2 = \&c. = 0$, the coefficient of $f^{(p)}a$ will vanish if $p > n$. And whatever the values of z, the coefficient of $f^{(p)}a$ vanishes if $p < n$: for we know that

$$\Sigma \frac{z_1^k}{(z_2 - z_1)\ldots.(z_n - z_1)} = 0,$$

k being $< n$, and $= 1$ when $k = n$;

$$\therefore A = \frac{f^{(n)}a}{1.2\ldots.n},$$

which was to be found.

(ϵ).